Generalisations of the Wielandt subgroup

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Chapter 1

Introduction

In this chapter we exhibit an overview on our results. In Sections 1.1 and 1.2 we first recall some of the well-known definitions and results on the norm, the Wielandt subgroup and the local Wielandt subgroup.
In Section 1.3 we introduce the generalised Wielandt subgroup. This has been defined and investigated in my MPhil Thesis [2]. The results of this thesis are published in my joint research paper [3]. We recall a variation of the results of [3] here for completeness and we combine this with some new results on the structure of the generalised Wielandt subgroup and also correct a mistake in [3]. Further, in Sections 1.4 and 1.5 we recall some well-known results on the Wielandt series and on T-groups together with some results on their variations for the generalised Wielandt subgroup; these results are also partially contained in [2] and [3].
In Sections 1.6 - 1.8 we then exhibit a variety of new results on the Wielandt subgroup and its generalisations. In Section 1.6 we introduce and investigate the relative Wielandt subgroup. In Section 1.7 we discuss algorithms to compute the Wielandt subgroup and its variations and, based on this, in Section 1.8 we consider the groups of order dividing p^6 with maximal Wielandt subgroup.

1.1 The norm and the Wielandt subgroup

In this section we recall some of the well-known basic definitions which we will use throughout, see also [29, page 398 ff].
The Wielandt subgroup and its generalisations are group theoretic concepts which have first been introduced as generalisations of the center of a group. Historically, a first idea in this direction is due to Baer (1935). We recall this as follows.

1.1. Definition: Let G be a group. The *Norm* $N(G)$ of G is the set of elements that normalises every subgroup of G; that is,

$$N(G) = \bigcap_{U \leq G} N_G(U).$$

By construction, the norm is a characteristic subgroup of G containing the center $Z(G)$ of G. It is closely related to the center of the group G as the following well-known theorem asserts. With $Z_2(G)$ we denote the second term of the upper central series of a group G; that is, $Z_2(G)/Z(G) = Z(G/Z(G))$.

1.2. Theorem: *Let G be a group.*

(i) (Schenkman; [32, Theorem 1]) $N(G) \leq Z_2(G)$.
(ii) (Baer; [5, Theorem 1]) *If G contains an element of infinite order, then $N(G) = Z(G)$.*
(iii) (Baer; [6, Theorem 1]) $N(G) = \{1\}$ *if and only if* $Z(G) = \{1\}$.

Hence the norm of a group is always rather close to the center of the group. This may be a reason why generalisations of Baer's concept have been considered. One of these generalisations is due to Wielandt (1958). Recall that a subgroup U of a group G is subnormal in G if there exists a series of subgroups $U = U_0 \leq U_1 \leq \ldots \leq U_l = G$ so that U_i is normal in U_{i+1}.

1.3. Definition: Let G be a group. The Wielandt subgroup $w(G)$ of G is the intersection of the normalisers of the subnormal subgroups of G; that is,

$$w(G) = \bigcap_{U \trianglelefteq\trianglelefteq G} N_G(U).$$

Thus by construction, the Wielandt subgroup is a characteristic subgroup of G containing the norm and thus the center of G. Wielandt [34] initiated a further investigation of the Wielandt subgroup and its properties. To recall some of the results, note that a group G satisfies the minimal condition on subnormal subgroups if every strictly descending chain of subnormal subgroups of G has finite length; we say that G satisfies Min-sn in this case. The minimal condition on normal subgroups is defined similarly and denoted with Min-n. Note that every finite group satisfies Min-sn and Min-n.

1.4. Theorem:

(i) (Wielandt; [34, Theorem 1(b)]) *Let N be a minimal normal subgroup of the group G so that N satisfies Min-n. Then $N \leq w(G)$.*
(ii) (Robinson & Roseblade; [29, Theorem 13.3.8]) *Let G be a group satisfying Min-sn. Then $w(G)$ has finite index in G.*

In particular, if G is a non-trivial finite group, then $w(G)$ is non-trivial, since contains every minimal normal subgroup of G and thus the socle $Soc(G)$ of G. Hence the Wielandt subgroup differs significantly from the center and the norm of its underlying group. On the other hand, there are groups with trivial Wielandt subgroup. We exhibit some examples to illustrate this.

1.5. Example:

(i) If G is finite non-abelian simple, then $\{1\} = Z(G) = N(G)$ and $w(G) = G$.
(ii) If G is nilpotent, then $\{1\} \lneq Z(G) \leq N(G) = w(G) \leq Z_2(G)$.
(iii) If G is the infinite dihedral group, then $\{1\} = Z(G) = N(G) = w(G)$.
(v) If $G = Q_{16} \times S_3$, then $\{1\} \lneq Z(G) \lneq N(G) \lneq w(G)$ with $Z(G) \cong C_2$, $N(G) \cong C_4$ and $w(G) \cong C_4 \times S_3$.

The Wielandt subgroup of a polycyclic group has various interesting properties. Lennox and Stonehewer [23] proved that $w(G)$ is either finite or abelian for a polycyclic group G. Cossey used this to show the following.

1.6. Theorem: (Cossey; [12, Theorem 1])

(i) *Let G be a polycyclic group. Then $G/C_G(w(G))$ is finite.*
(ii) *Let G be nilpotent-by-abelian. Then $w(G)/Z(G)$ is finite.*

Hence it seems that in a polycyclic group the Wielandt subgroup is close to the center. On the other hand, there are nilpotent-by-finite groups G where $w(G)/Z(G)$ is non-trivial free abelian.

1.2 The local Wielandt subgroup

The idea of the Wielandt subgroup was generalised further. For example, Bryce & Cossey [8] introduced the following local version of it. For a prime p a group G is called p'-perfect if every non-trivial factor group G/N of G satisfies that $p \mid |G/N|$.

1.7. Definition: Let G be a group and p a prime. Then local Wielandt subgroup $w^p(G)$ is the intersection of the normalisers of the p'-perfect subnormal subgroups of G; that is,

$$w^p(G) = \bigcap_{\substack{U \lhd\lhd G \\ U p'\text{-perfect}}} N_G(U).$$

Then the local Wielandt subgroup is a characteristic subgroup of G which contains the Wielandt subgroup. A main result about the local Wielandt subgroup is the following.

1.8. Theorem: (Bryce & Cossey; [8, Theorem 3.7])
Let G be a finite soluble group. Then

$$w(G) = \bigcap_{p\in\pi} w^p(G).$$

1.3 The generalised Wielandt subgroup

A different generalisation of the Wielandt subgroup has been introduced in my joint research paper with Ali & Arif [3] which is based on my MPhil Thesis [2]. We recall and extend these results here in this thesis. In this section we first give an overview of this.
As a first step, we recall the definition of the generalised Wielandt subgroup as follows.

1.9. Definition: Let G be a group and N a normal subgroup of G. The generalised Wielandt subgroup $w_N(G)$ of G with respect to N is the intersection of the normalisers of the subnormal subgroups of G contained in N; that is,

$$w_N(G) = \bigcap_{\substack{U \trianglelefteq\trianglelefteq G \\ U\leq N}} N_G(U)$$

If N is characteristic in G, then $w_N(G)$ is characteristic in G. Further, $w(G) \leq w_N(G)$ for every normal subgroup N of G and

$$w(G) = \bigcap_{N\trianglelefteq G} w_N(G).$$

As a first result, we prove the following for finite nilpotent groups, see Section 2.1, see also [3].

1.10. Theorem: *Let G be finite nilpotent so that G is the direct product of its Sylow subgroups: $G \cong P_1 \times \ldots \times P_l$. Then*

$$w(G) = \bigcap_{i=1}^{l} w_{P_i}(G).$$

Next, we generalise Theorem 1.4(i) of Wielandt [34] as follows. The following result is also proved in [3]. We give a new proof in Section 2.1.

1.11. Theorem: *Let G be a group and $N \trianglelefteq G$. Then*

(i) *$w_N(G)$ contains every simple non-abelian subnormal subgroup of N.*

(ii) *$w_N(G)$ contains every minimal normal subgroup K of N such that K satisfies Min-sn.*

Next, we consider the generalised Wielandt subgroup in groups which are semidirect products. We prove the following in Section 2.2, see also [3].

1.12. Theorem: *Let $G = A \rtimes B$.*

(i) *If N is a normal subgroup of G with $N \leq A$, then $w_N(G) \cap A = w_N(A)$.*

(ii) *Suppose that A and B have coprime order and A is nilpotent. If N is a normal subgroup of G with $A \leq N$ and P is the set of those elements of $w_{N\cap B}(B)$ which induce power automorphisms by conjugation on A, then $w_N(G) = Pw(A)$.*

The generalised Wielandt subgroup also has a local version. If G is a group and N is a normal subgroup of G, then the local generalised Wielandt subgroup $w_N^p(G)$ of G with respect to N is the intersection of the normalisers of the p'-perfect subnormal subgroups of G contained in N; that is

$$w_N^p(G) = \bigcap_{\substack{U \trianglelefteq\trianglelefteq G \\ U \leq N \\ U p'\text{-perfect}}} N_G(U).$$

We generalise the result of Bryce & Cossey [8] on local Wielandt subgroups to local generalised Wielandt subgroups. See Section 2.3 for a proof, see also [3].

1.13. Theorem: *Let G be a finite soluble group and $N \trianglelefteq G$. If π is the set of primes dividing $|G|$, then*

$$w_N(G) = \bigcap_{p\in\pi} w_N^p(G).$$

By using Theorem 1.13 we obtain the following relation between the generalised Wielandt subgroup and the local generalised Wielandt subgroup.

1.14. Theorem: *Let G be a finite soluble group and $M, N \trianglelefteq G$.*

(i) *If G/N is a p'-group, then $O_p(w^p(N)) = O_p(w_N(G))$.*

(ii) *Let p be a prime and suppose that $O_{p'}(G) \leq M$. Then*

$$w_{M/O_{p'}(G)}(G/O_{p'}(G)) = w_M^p(G)/O_{p'}(G).$$

1.4 The Wielandt series and its generalisation

Based on the Wielandt subgroup, Wielandt [34] defined a series

$$\{1\} = w_0(G) \leq w_1(G) \leq w_2(G) \leq \ldots$$

for a group G inductively by $w_{i+1}(G)/w_i(G) = w(G/w_i(G))$. The least n for which $w_n(G) = G$, if it exists, is called the Wielandt length of G and denoted $wl(G)$. Some major results concerning the relationship between the Wielandt length and other invariants have been exhibited by Camina [10] and by Bryce & Cossey [8].

1.15. Theorem: *Let G be a finite group with $n = wl(G)$. Then*

(i) (Camina; [10, Proposition 1])
$l(G) \leq n+1$.

(ii) (Bryce & Cossey; [8, Theorem 4.1])

$$d(G) \leq \begin{cases} 5m & \text{if } n = 3m \\ 5m+2 & \text{if } n=3m+1 \\ 5m+4 & \text{if } n=3m+2 \end{cases}.$$

Here we introduce a similar series for the generalised Wielandt subgroup and investigate the properties of this series, see also [3]. For a group G and $N \trianglelefteq G$ let

$$\{1\} = w_{N,0}(G) \leq w_{N,1}(G) \leq w_{N,2}(G) \leq \ldots$$

where $w_{N,1}(G) = w_N(G)$ and, inductively, if $\varphi_i : G \to G/w_{N,i}(G)$ is the natural epimorphism on the quotient group, then $w_{N,i+1}(G)$ is the full preimage of $w_{\varphi_i(N)}(\varphi_i(G))$ under φ_i; that is

$$w_{N,i+1}(G)/w_{N,i}(G) = w_{\varphi_i(N)}(\varphi_i(G)).$$

If G is a finite group, then there exists an n with $w_{N,n}(G) = G$. The smallest such n is called the generalised Wielandt length $wl_N(G)$ of G with respect to N. In Section 3.1 we prove the following bounds on the generalised Wielandt length in supersoluble groups, see also [3].

1.16. Theorem: *Let G be a supersoluble group and let $N \trianglelefteq G$.*

(i) *If all Sylow p-subgroups of G for odd p are abelian, and if the Sylow 2-subgroups have class at most two, then $wl_N(G) \leq 2$.*

(ii) *If n is the maximum of the nilpotency classes of Sylow p-subgroups of G which contain N, then $wl_N(G) \leq n+1$.*

Next, we prove the following new alternative characterisation of the higher order generalised Wielandt subgroups, see Section 3.2 for proof.

1.17. Theorem: *Let G be a finite group and $N \trianglelefteq G$. Then for $i \geqslant 0$,*

$$w_{N,i+1}(G) = \cap\{N_G(K) \mid w_{N,i}(G) \leq K \leq N w_{N,i}(G), K \lhd\lhd G\}.$$

Based on Theorem 1.17 we then obtain the following new result, see Section 3.3 for a proof.

1.18. Theorem: *Let G be a finite group, let $N \trianglelefteq G$ and let $m, n \geq 0$ be integers. If the generalised Wielandt length of $G/w_{N,m}(G)$ with respect to $Nw_{N,m}(G)/w_{N,m}(G)$ is n, then $wl_N(G) = m + n$.*

1.5 T-groups and its generalisation

A group G is called T-group if it has Wielandt length one; that is, if $w(G) = G$ or, equivalently, if every subnormal subgroup of G is normal. The Wielandt subgroup of a finite group G is always a T-group. The structure of T-groups was first investigated by Zacher [35]. Further investigations are due to Gaschütz [18], Peng [27] and Robinson [28]. The group in which all cyclic subnormal subgroups are normal was introduce by Sakamoto in [31] and denoted T_c-group.
Here we combine the concept of generalised Wielandt subgroup and T-groups and investigate the groups G and the normal subgroups N with $wl_N(G) = 1$. We prove the following in Section 3.4.

1.19. Theorem: *Let G be a group and $N, M \trianglelefteq G$ with $M \leq N$. If $wl_N(G) = 1$, then $wl_{N/M}(G/M) = 1$.*

Further, we show the following in Section 3.4.

1.20. Theorem: *Let G be a group. If $wl_{\mathrm{Fit}(G)}(G) = 1$, then $C_G(G') = Fit(G)$ and $\mathrm{F}it(G)$ is a Dedekind group.*

1.6 The relative Wielandt subgroup and its series

We define a new generalisation "relative Wielandt subgroup" of the Wielandt subgroup and denoted it by $w_c(G)$.

1.21. Definition: Let G be a group. The relative Wielandt subgroup $w_c(G)$ of G is the intersection of the normalisers of the cyclic subnormal subgroups of G; that is,

$$w_c(G) = \bigcap_{\substack{U \lhd\lhd G \\ U \text{is cyclic}}} N_G(U).$$

Thus by construction, $w_c(G)$ is a characteristic subgroup of G. It follows from the definition of the relative Wielandt subgroup that $w(G) \leq w_c(G)$ hold for every group G. Thus $w_c(G)$ is always non-trivial for a finite non-trivial group G.

The relative Wielandt series

$$\{1\} = w_{0,c}(G) \leq w_{1,c}(G) \leq w_{2,c}(G) \leq ...$$

where $w_{1,c}(G) = w_c(G)$, for a group G inductively defined by $w_{i+1,c}(G)/w_{i,c}(G) = w_c(G/w_{i,c}(G))$. The least n for which $w_{n,c}(G) = G$, if it exists, is called the relative Wielandt length of G and denoted by $wl_c(G)$. Since $w_c(G) \neq \{1\}$ for a finite group $G \neq \{1\}$, consequently the relative Wielandt series is well defined.
Further we define the local relative Wielandt subgroup of a group G as follows;

1.22. Definition: Let G be a group and p a prime. Then the local relative Wielandt subgroup $w_c^p(G)$ is the intersection of the normalisers of the cyclic p'-perfect subnormal subgroups of G; that is,

$$w_c^p(G) = \bigcap_{\substack{U \lhd\lhd G \\ U p'\text{-perfect} \\ U \text{is cyclic}}} N_G(U).$$

We prove the following for the relative Wielandt subgroup, for proof see Section 4.1.

1.23. Theorem: *If G is a quasisimple group with cyclic center, then $w_c(G) = w(G)$.*

The next theorem does not hold for the Wielandt subgroup.

1.24. Theorem: *Let G and H be a groups. Then*

(i) $w_c(G \times H) = w_c(G) \times w_c(H)$.
(ii) *If $G = D_\infty$, then $w_c(G) = G$.*

1.25. Theorem: *Let G be a polycyclic group. Then*

(i) $w_c(G)Z(Fit(G))/Z(Fit(G)) \cong w_c(G)/w_c(G) \cap Z(Fit(G))$ *is finite abelian.*
(ii) $w_c(G)$ *is supersoluble and metabelian.*

The next theorem is an another generalisation of the Theorem 1.8, see Section 4.2 for proof.

1.26. Theorem: *Let G be a finite group and if π is the set of primes dividing $|G|$. Then*

$$w_c(G) = \bigcap_{p \in \pi} w_c^p(G).$$

1.7 Algorithms to compute Norm and generalised Wielandt subgroup

We developed algorithms to compute the Norm of a group G for a finite or a polycyclic group G. We also introduced methods to determine the ordinary or generalised or relative Wielandt subgroup of a finite group G. We note that there is no method available to compute the Wielandt subgroup of an arbitrary infinite polycyclic group at current; so this is an open problem. Further, for finite groups, we give a method to check that whether the given group is a T-group or not. We exhibit all of these algorithms in Sections 5.1, 5.2 and 5.3 and we include various sample applications.

1.8 Groups of order dividing p^6 with large Wielandt subgroup

A finite p-group G is nilpotent and hence $Z(G) \leq w(G) = N(G) \leq Z_2(G)$ follows. Thus it seems interesting to try to understand the extremal case; that is, those p-groups G with $w(G) = Z_2(G)$. We obtain the following. See Section 6.1 for a proof.

1.27. Theorem: *Let G be a finite p-group with $w(G) = Z_2(G)$. Then*

$$w(G/Z(G)) = Z(G/Z(G)).$$

1.28. Theorem: *Let G be a finite p-group with $Exp(G) = p$. Then $N(G) = w(G) = Z(G)$.*

A library of groups of order dividing p^6 has been determined by Newman, O'Brien and Vaughan-Lee [24]. The results of this classification are available in the LiePRing Package [33] of GAP [17]. We use this and determine the groups G of order dividing p^4 with $w(G) = Z_2(G)$. Further, we give a conjectural description of the groups of order p^5 and p^6 with $w(G) = Z_2(G)$. We exhibit this result in the Appendix A.

Chapter 2

The generalised Wielandt subgroup

The aim of this chapter is to investigate properties of the generalised Wielandt subgroup.

2.1 General investigations

In this section we exhibit some basic properties of the generalised Wielandt subgroup. The first remark exhibits a condition under which the generalised Wielandt subgroup is the full group.

2.1. Remark: Let G be a group. Then $w_N(G) = G$ if N is a cyclic normal subgroup of G.

Next, we investigate conditions under which the generalised Wielandt subgroup and the Wielandt subgroup coincide.

2.2. Lemma: *Let G be a group. Then $w_N(G) = w(G)$ if $N = G$ or N is the unique maximal normal subgroup of G.*

Proof: The result is obvious for $N = G$. Thus let N be the unique maximal normal subgroup of G. Then every subnormal subgroup of G is contained in N and hence $w_N(G) \subseteq w(G)$. In the other direction it follows from the definition of $w_N(G)$ that $w(G) \subseteq w_N(G)$ hold for every group G. Hence the result follows. •

2.3. Example: We exhibit some elementary examples.

(i) $w_{A_3}(S_3) = S_3 = w(S_3)$.
(ii) $w_{A_4}(S_4) = w_{V_4}(S_4) = w(S_4) = V_4$.
(iii) $w(S_n) = S_n = w_{A_n}(S_n)$ for $n \geqslant 5$.
(iv) $w(D_{2^n}) = Z(D_{2^n})$ and $w_{D_{2^n}}(D_{2^n}) = D_{2^n}$ for $n \geqslant 1$.

(v) Let $G = Q_{16} \times S_4$ and $N \cong Q_8 \times S_4 \trianglelefteq G$. Then $w(G) \cong C_4 \times V_4$ and $w_N(G) \cong Q_8 \times V_4$.

Next, we exhibit an elementary lemma on the generalised Wielandt subgroup which will be useful in the proofs of many of the later results. The proof is straightforward, and we omit it.

2.4. Lemma: *Let G be a group.*

(i) *If $M, N \trianglelefteq G$ with $M \subseteq N$, then $w_N(G) \subseteq w_M(G)$.*
(ii) *If $M \leq G$ and $N \trianglelefteq G$ with $N \subseteq M$, then $w_N(G) \cap M = w_N(M)$.*
(iii) *If G is a T-group and $N \trianglelefteq G$, then $w_N(G) = G$.*
(iv) *If $N \trianglelefteq G$, then $w(G)N/N \subseteq w_N(G)N/N$.*

It is well-known that $w(G)N/N \leq w(G/N)$ for any group G and any normal subgroup N of G, see Lemma 3.1(i) of [8]. Further, Lemma 2.4(iv) asserts that $w(G)N/N \subseteq w_N(G)N/N$ holds. We note that $w(G/N) \nsubseteq w_N(G)N/N$ in general using the following example.

2.5. Example: Let G be a group of order 2^8 described by the finite presentation on generators $g_1, \ldots, g_8$ with relations of the form

$$g_1^2 = g_4, g_2^2 = g_5, g_3^2 = g_6, g_4^2 = g_7, g_5^2 = g_8, [g_2, g_1] = g_3, [g_3, g_1] = g_6, [g_5, g_1] = g_6.$$

Further, let $N = \Phi(G)$ the Frattini subgroup of G. Then $w_N(G) = \langle g_2, g_3, g_4, g_5, g_6, g_7, g_8 \rangle$ and $w(G/N) = G/N$. Thus $w_N(G)N/N$ has order 2 and $w(G/N)$ has order 4, so that $w(G/N) \nleq w_N(G)N/N$ follows.

We now consider the relationship between the generalised Wielandt subgroups for different normal subgroups in more detail.

2.6. Theorem: *Let G be a group and $M, N \trianglelefteq G$. If $(|M|, |N|) = 1$, then $w_{MN}(G) = w_M(G) \cap w_N(G)$.*

Proof: "$\supseteq$" Let g be an arbitrary element of $w_M(G) \cap w_N(G)$ and S be a subnormal subgroup of G contained in MN. As M and N are of coprime order, it follows from [1, Lemma 2.1] that $S = (S \cap M)(S \cap N)$. Now consider

$$\begin{aligned} S^g &= ((S \cap M)(S \cap N))^g \\ &= (S \cap M)^g (S \cap N)^g \\ &= (S \cap M)(S \cap N) \\ &= S. \end{aligned}$$

Therefore g belongs to $w_{MN}(G)$. Hence $w_M(G) \cap w_N(G) \subseteq w_{MN}(G)$.

"$\subseteq$" Since $M \subseteq MN$ and $N \subseteq MN$, Remark 2.4(i) asserts that $w_{MN}(G) \subseteq w_M(G)$ and $w_{MN}(G) \subseteq w_N(G)$. Thus $w_{MN}(G) \subseteq w_M(G) \cap w_N(G)$ and the result follows. •

We observe that if $M \cap N = 1$ instead of $(|M|, |N|) = 1$ in Theorem 2.6, then $w_M(G) \cap w_N(G) \subseteq w_{M\cap N}(G)$ and the equality will not holds in general.

2.7. Example: Let G be a group of order 3^6 with generators $g_1, \dots, g_6$ and relations

$$g_1^3 = g_2, g_2^3 = g_3, g_4^2 = g_6, g_5^2 = g_6, g_4^{g_1} = g_5, g_5^{g_1} = g_4g_5, g_5^{g_4} = g_5g_6.$$

Let $N = \langle g_4, g_5, g_6\rangle$ and $M = \langle g_3\rangle \trianglelefteq G$. Note that $M \cap N = \{1\}$. Thus $w_N(G) = \langle g_2, g_3, g_4, g_5, g_6\rangle$ and $w_M(G) = G$ and $w_{M\cap N}(G) = G$. Hence

$$w_{M\cap N}(G) \neq w_M(G) \cap w_N(G).$$

The following is an immediate corollary of Theorem 2.6, see also Theorem 1.10.

2.8. Corollary: *Let G be finite nilpotent so that G is the direct product of its Sylow subgroups: $G \cong P_1 \times \ldots \times P_l$. Then*

$$w(G) = \bigcap_{i=1}^{l} w_{P_i}(G).$$

In the following proposition we characterise the normal subgroups N which are contained in $w_N(G)$, see also [3].

2.9. Proposition: *Let G be a group and $N \trianglelefteq G$. Then N is a T-group if and only if N is contained in $w_N(G)$.*

Proof: "$\Rightarrow$" Let us suppose that N is a T-group and let $n \in N$. Let S be a subnormal subgroup of G contained in N. Then S is subnormal in N and, since N is a T-group, it follows that S is normal in N. Thus $S^n = S$ and n belongs to $w_N(G)$. In summary, $N \leq w_N(G)$.
"$\Leftarrow$" We suppose that $N \leq w_N(G)$. Let S be a subnormal subgroup of N. Since N is normal in G, it follows that S is subnormal in G. Thus $S^g = S$ for all $g \in w_N(G)$. Hence $S^n = S$ for all $n \in N$ and $S \trianglelefteq N$. This yields the desired result. •

The following is the first main theorem of this chapter, see also Theorem 1.11.

2.10. Theorem: *Let G be a group and $N \trianglelefteq G$. Then*

(i) *$w_N(G)$ contains every simple non-abelian subnormal subgroup of N.*
(ii) *$w_N(G)$ contains every minimal normal subgroup K of N such that K satisfies Min-sn.*

Proof: (i) Since $w(G) \subseteq w_N(G)$ and every subnormal subgroup of N is also subnormal in G, the result follows from Wielandt's result [34] that $w(G)$ contains every simple non-abelian subnormal subgroup of G.
(ii) Let H be a subnormal subgroup of G such that $H \subseteq N$. Then H is subnormal in N. Let n be the length of a shortest subnormal series from H to N. We proceed by induction on n. If $n = 1$, then $H \trianglelefteq N$ and by the minimality of K, it follows that $K \subseteq N_N(H)$. Hence $K \subseteq N_G(H)$. Now let $n > 1$. By the minimality of K, we have either $H^N \cap K = 1$ or $K \subseteq H^N$. If $H^N \cap K = 1$, then $[H, K] \subseteq H^N \cap K = 1$ follows and this implies that $K \subseteq N_N(H)$. Hence $K \subseteq N_G(H)$. Now consider the second case $K \subseteq H^N$. Let M be a minimal normal subgroup of H^N contained in K. Then each conjugate of M in N is a minimal normal subgroup of H^N and satisfies Min-sn. Hence by the induction hypothesis, each conjugate of M normalises H. Therefore $M^N \subseteq N_N(H)$. As $M^N = K$, this yields that $K \subseteq N_N(H)$ and the result follows. •

The following remark is a direct consequence of Theorem 2.10.

2.11. Remark:

(i) Let $N \trianglelefteq G$ and let K be a minimal normal subgroup of N. If K is not normal in G, then by [34, Theorem 1(b)] it is possible that $K \nsubseteq w(G)$.
(ii) If $N \trianglelefteq G$ and every minimal normal subgroup of N satisfies Min-sn, then $w_N(G)$ contains the socle of N.

D. J. S. Robinson has pointed out that Theorem [3, Theorem 2.7] is incorrect. The following observation provides an alternative to this theorem.

2.12. Theorem: *Let G be a group satisfying Min-sn and let $N \trianglelefteq G$. Then $w_N(G)$ has finite index in G.*

Proof: By [29, Theorem 13.3.8] we obtain that $[G : w(G)]$ is finite. Thus the result follows from $w(G) \leq w_N(G)$. •

We show that $w_N(G)$ can have infinite index in G.

2.13. Example: Let N be the direct product of two copies of a Prüfer p^∞-group A and let x be the automorphism of N defined by $A \times A \to A \times A : (a, b)^x = (a, a^p b)$. Let $G = N \rtimes \langle x \rangle$. Then N has infinite index in G and satisfies minimal conditions on subnormal subgroups, but G does not satisfy Min-sn. Let $H = \{(a, a) | a \in A\}$. Then H is a subnormal subgroup of G such that $H \subset N$. As $H \trianglelefteq N$, it follows that $N_G(H) = N$ and hence $w_N(G) = N$. Thus $w_N(G)$ has infinite index in G.

2.14. Remark: Let $x \in w_N(G) \setminus w(G)$ and $S \lhd\lhd G$, such that $S^x \neq S$. Then S does not occur in any subnormal series of G in which N occurs.

2.15. Example: Let $G = D_8 = \langle (1234), (24) \rangle$ and $N = \langle (1234) \rangle$ a normal subgroup of G. Then $w(G) = \langle (13)(24) \rangle = Z(G)$ and $w_N(G) = G$. The subnormal subgroups of G with $S^x \neq S$ for some $x \in w_N(G) \setminus w(G)$ are $S_1 = \langle (24) \rangle$, $S_2 = \langle (13) \rangle$, $S_3 = \langle (12)(34) \rangle$ and $S_4 = \langle (14)(23) \rangle$. Hence $N \nsubseteq S_i$ for $i \in \{1, 2, 3, 4\}$.

2.2 Semidirect products

The technique for calculating the Wielandt subgroup of a semidirect product of two groups of coprime order was developed by Ali [1]. Here we generalise this to the generalised Wielandt subgroup, see also [3].

2.16. Lemma: *Let $G = A \rtimes B$. If $N \trianglelefteq G$ with $A \subseteq N$, then*

$$w_N(G) \cap B \subseteq w_{N\cap B}(B).$$

Proof: Let S be a subnormal subgroup of B contained in $N \cap B$. Then SA is subnormal in G and $SA \subseteq N$. Let x be an element of $w_N(G) \cap B$. Then $(SA)^x = SA$, which implies that $S^x A = SA$. Now for each $s \in S$ there is some $s_1 \in S$ so that $s^x \in s_1 A$. Hence $s_1^{-1} s^x \in A \cap B = 1$. This implies that $s^x = s_1 \in S$ for all $x \in w_N(G) \cap B$. Hence $w_N(G) \cap B \subseteq w_{N\cap B}(B)$. •

2.17. Lemma: *Let $G = A \rtimes B$. If N is a normal subgroup of G such that A is the unique maximal normal subgroup of N, then*

$$w_N(G) \cap A = w(A).$$

Proof: "$\subseteq$" Let x be an element of $w_N(G) \cap A$ and let S be a subnormal subgroup of A. Then S is subnormal in G and contained in N. Therefore $S^x = S$ and x belongs to $w(A)$. Thus $w_N(G) \cap A \subseteq w(A)$.
"$\supseteq$" Let x be an element of $w(A)$ and let S be a subnormal subgroup of G contained in N. Since A is the unique maximal normal subgroup of N, hence S is subnormal in A as well. Therefore $S^x = S$ and hence x belongs to $w_N(G) \cap A$. Hence the result follows. •

The following is an easy corollary of Lemma 2.17.

2.18. Corollary: *Let $G = A \rtimes B$. If $N \trianglelefteq G$ and $N \leq A$, then*

$$w_N(G) \cap A = w_N(A).$$

Now we use the results Lemmas 2.16 and 2.17 to obtain the following theorem, see Theorem 1.12.

2.19. Theorem: *Let $G = A \rtimes B$, where A and B have coprime order and A is nilpotent. If N is a normal subgroup of G with $A \leq N$ and P is the set of those elements of $w_{N\cap B}(B)$ which induce power automorphisms by conjugation on A, then*

$$w_N(G) = Pw(A).$$

Proof: First, since $w_N(G)$ is normal in G it is subnormal in G. By [1, Lemma 2.1] every subnormal subgroup S of $G = BA$ with $(|A|, |B|) = 1$ can be written as

$$S = (S \cap B)(S \cap A).$$

Therefore we obtain that

$$w_N(G) = (w_N(G) \cap B)(w_N(G) \cap A).$$

By Lemma 2.17, it follows that

$$w_N(G) = (w_N(G) \cap B)w(A).$$

Since A is normal and nilpotent, every subgroup of A is subnormal in G. Further, as A is contained in N, it follows that $w_N(G)$ normalises all subgroups of A. Hence $w_N(G) \cap B \subseteq P$. It remains to prove that $P \subseteq w_N(G) \cap B$. Since $P \subseteq B$ by definition of P, it will be enough to show that $P \subseteq w_N(G)$. For this let S be a subnormal subgroup of G such that $S \subseteq N$. By [1, Lemma 2.1] it follows that $S = (S \cap B)(S \cap A)$. Since $S \cap B$ is subnormal in B and $S \cap B \subseteq N \cap B$, therefore normalised by $w_{N\cap B}(B)$ and hence is normalised by P. Furthermore, it follows directly from the definition of P, that P normalises $S \cap A$. Hence P normalises S. This yields the result. •

2.3 The local generalised Wielandt subgroup

We now consider the local generalised Wielandt subgroup as defined in Section 1.3. In particular, we establish a relationship between the local generalised Wielandt subgroup and the generalised Wielandt subgroup. Parts of this section also appeared in [3].

2.20. Remark: Let G be a group, $N \trianglelefteq G$ and p a prime. Then

(i) $w(G) \leq w_N(G) \leq w_N^p(G)$.

(ii) $w(G) \leq w^p(G) \leq w_N^p(G)$.

We introduce some further notation. Let π be a set of primes. Then a π-number is a positive integer whose prime divisors all belong to π. An element of a group G is known as π-element if its order is a π-number. If every element of a group is a π-element, then the group itself is a π-group. We note that for a prime p we write p' to denote the set of primes other than p. Recall that the subgroup generated by all normal π-subgroups of a group G is denoted by $O_\pi(G)$. It is the unique maximal normal π-subgroup of G.

2.21. Theorem: *Let G be a finite soluble group and $M, N \trianglelefteq G$. Then*

(i) $w_{MN}(G)N/N \subseteq w_{MN/N}(G/N)$ *and* $w^p_{MN}(G)N/N \subseteq w^p_{MN/N}(G/N)$.

(ii) $w_N(G) \cap N = w(N)$ *and* $w^p_N(G) \cap N = w^p(N)$.

(iii) *If N is a p'-group, then* $N \subseteq w^p_{MN}(G)$ *and* $w^p_{MN/N}(G/N) = w^p_{MN}(G)/N$.

(iv) $O_{p'}(G) \subseteq w^p_N(G)$ *and* $Soc(G/O_{p'}(G)) \subseteq w^p_{NO_{p'}(G)}(G)/O_{p'}(G)$.

(v) *If G/N is a p'-group, then*

$$O_p(w^p(N)) = O_p(w^p_N(G)).$$

Proof: (i) Let g be an element of $w_{MN}(G)N/N$. Write $g = xN$ for some $x \in w_{MN}(G)$. Let S/N be a subnormal subgroup of G/N contained in MN/N. Hence S is a subnormal subgroup of G and contained in MN. Now consider

$$\begin{aligned}(S/N)^g &= g^{-1}(S/N)g\\ &= (xN)^{-1}(S/N)(xN)\\ &= (x^{-1}N)(S/N)(xN)\\ &= x^{-1}Sx/N = S^x/N\\ &= S/N.\end{aligned}$$

Thus $g \in w_{MN/N}(G/N)$ and hence $w_{MN}(G)N/N \subseteq w_{MN/N}(G/N)$. The proof of the second part of the statement can be obtain in a similar form: Just replace subnormal subgroup by p'-perfect subnormal subgroup.
(ii) This follows directly from the definitions of $w_N(G)$ and $w^p_N(G)$.
(iii) By (i) it is sufficient to show that $w^p_{MN/N}(G/N) \subseteq w^p_{MN}(G)/N$. We prove this by induction on $|N|$. Let N be a minimal normal subgroup of G. Then $N \subseteq w^p(G)$. Since $w^p(G) \subseteq w^p_{MN}(G)$, we obtain the desired result. Let xN be an arbitrary element of $w^p_{MN/N}(G/N)$. Further, let S be a p'-perfect subnormal subgroup of G contained in MN. Then $x \in N_G(NS)$ and hence by [8, Lemma 2.2(i)] $x \in N_G(S)$. Therefore $xN \in w^p_{MN}(G)/N$. Let $N_1 \leq N$ be a minimal normal subgroup of G. Then by induction $N/N_1 \subseteq w^p_{MN/N_1}(G/N_1) = w^p_{MN}(G)/N_1$. Hence $N \subseteq w^p_{MN}(G)$. Let

$$\alpha : G/N \to (G/N_1)/(N/N_1)$$

be the natural isomorphism, then

$$\begin{aligned}(w^p_{MN/N}(G/N))\alpha &= w^p_{(MN/N_1)/(N/N_1)}((G/N_1)/(N/N_1))\\ &= w^p_{MN/N_1}(G/N_1)/(N/N_1), \text{by induction}\\ &= (w^p_{MN}(G)/N_1)/(N/N_1), \text{by induction}\\ &= (w^p_{MN}(G)/N)\alpha.\end{aligned}$$

Therefore $w^p_{MN/N}(G/N) = w^p_{MN}(G)/N$ follows.
(iv) Its follows from (iii) that $O_{p'}(G) \subseteq w^p_N(G)$. Since

$$Soc(G/O_{p'}(G)) \subseteq w_{NO_{p'}(G)/O_{p'}(G)}(G/O_{p'}(G)) \subseteq w^p_{NO_{p'}(G)/O_{p'}(G)}(G/O_{p'}(G)),$$

the result follows by using (iii).
(v) As G/N is a p'-group, N contains every p-subgroup of G. Hence $O_p(w^p_N(G)) \subseteq N$. Now by using (ii) and [30, Theorem 157(i)] it follows that

$$\begin{aligned}O_p(w^p_N(G)) &= N \cap O_p(w^p_N(G))\\ &= N \cap w^p_N(G) \cap O_p(G)\\ &= w^p(N) \cap O_p(G)\\ &= O_p(w^p(N)).\end{aligned}$$

•

We notice that the equality in Theorem 2.21(ii) does not holds for ordinary Wielandt subgroup see [8, Lemma 3.1].
The next theorem is the generalised form of the main theorem of Bryce & Cossey in [8], see also Theorem 1.13.

2.22. Theorem: *Let G be a finite soluble group and $N \trianglelefteq G$. If π is the set of all primes dividing $|G|$, then*

$$w_N(G) = \bigcap_{p\in\pi} w^p_N(G).$$

Proof: Remark 2.20(i) yields that $w_N(G) \subseteq w^p_N(G)$ for every prime p. It remains to show that $\cap_{p\in\pi} w^p_N(G) \subseteq w_N(G)$. Let S be a subnormal subgroup of G such that $S \subseteq N$. First suppose that S has a unique maximal normal subgroup M. Then S/M is cyclic of order p for some prime p. If S is not p'-perfect subnormal, then there exists a normal subgroup K of S with S/K a p'-group. This implies $K \subseteq M$ and so p must divide $|S/N|$, a contradiction. This proves the result in the case that S has a unique maximal normal subgroup. Now consider the arbitrary case on S. The group S can be written as a product of subnormal subgroups $S_1, \ldots, S_l$ each of which has a unique maximal normal subgroup. By the first case, each S_i is p'-perfect. Further, by the first case, each S_i is normalized by $\cap_{p\in\pi} w_N^p(G)$. Thus the result follows. •

In the following results we establish a relation between the generalised Wielandt subgroup and the local generalised Wielandt subgroup, see also Theorem 1.14.

2.23. Theorem: *Let G be a finite soluble group and $M, N \trianglelefteq G$.*
(i) *If G/N is a p'-group, then $O_p(w^p(N)) = O_p(w_N(G))$.*
(ii) *Let p be a prime and suppose that $O_{p'}(G) \leq M$. Then*

$$w_{M/O_{p'}(G)}(G/O_{p'}(G)) = w_M^p(G)/O_{p'}(G).$$

Proof: (i) "$\subseteq$" By Theorem 2.21(v), we have that $O_p(w^p(N)) = O_p(w_N^p(G))$. Therefore we only need to show that $O_p(w_N^p(G)) = O_p(w_N(G))$, since $O_p(w_N^p(G)) \subseteq w_N^p(G)$. Let $q \neq p$ be a prime. Then by Theorem 2.21(iii), we have that

$$O_p(w_N^p(G)) \subseteq O_{q'}(G) \subseteq w_N^q(G)$$

holds for all primes q. Hence by Theorem 2.22 it follows that $O_p(w_N^p(G)) \subseteq w_N(G)$ and thus [30, Theorem 157(i)] implies that

$$O_p(w_N^p(G)) \subseteq w_N(G) \cap O_p(G) = O_p(w_N(G)).$$

"$\supseteq$" The converse is obvious by Remark 2.20(i).
(ii) By Theorem 2.21(iii) we have that

$$w^p_{M/O_{p'}(G)}(G/O_{p'}(G)) = w_M^p(G)/O_{p'}(G).$$

Hence we can write

$$w_{M/O_{p'}(G)}(G/O_{p'}(G)) = w^p_{M/O_{p'}(G)}(G/O_{p'}(G)).$$

Let $A = O_{p'}(G)$. So by Theorem 2.21(iii), for convenience, we can assume $A = I$. Thus it will be sufficient to prove, $w_M(G) = w_M^p(G)$. The rest of the proof proceeds as in (i). •

2.24. Theorem: *Let G be a finite soluble group and $N \trianglelefteq G$. If G/N is a p'-group, then*
(i) $O_p(w_N(G)) = O_p(w_N^p(G))$.
(ii) $O_p(w_N(G)) \cap N = O_p(w(N))$.

Proof: (i) By Theorem 2.21(v), we have that $O_p(w^p(N)) = O_p(w_N^p(G))$ and by Theorem 2.23(i) $O_p(w^p(N)) = O_p(w_N(G))$. Hence

$$O_p(w_N(G)) = O_p(w_N^p(G)).$$

(ii) By Theorem 2.23(i) we have that $O_p(w_N(G)) = O_p(w^p(N))$. Hence

$$O_p(w_N(G)) \cap N = O_p(w^p(N)) \cap N = O_p(w^p(N)).$$

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Chapter 3

The generalised Wielandt length

A main aim of this chapter is to investigate the generalised Wielandt length. In particular, we determine a bound for the generalised Wielandt length of a supersoluble group. Further, we exhibit a characterisation for the higher order generalised Wielandt subgroups.
Every group G has a unique Wielandt subgroup $w(G)$. But there exists a one-to-one correspondence between the normal subgroups and the generalised Wielandt subgroups $w_N(G)$ of a group G with respect to N. In the following example we shows that the generalised Wielandt length of a group G with respect to any normal subgroup N of G is always less than or equal to the Wielandt length of G .

3.1. Example: Let G be a group of order 2^4 with generators $g_1, \ldots, g_4$ and relations

$$g_2^2 = g_3, g_3^2 = g_4, g_2^{g_1} = g_2g_3g_4, g_3^{g_1} = g_3g_4.$$

Then G has seven normal subgroups. The generalised Wielandt series with respect to each normal subgroup of G is as follows;

Normal subgroup	Generalised Wielandt series
$N_1 = G$	$\{1\} \leq \langle g_4 \rangle \leq \langle g_4, g_3 \rangle \leq G$
$N_2 = \langle g_1g_2g_3g_4, g_3, g_4 \rangle$	$\{1\} \leq \langle g_4 \rangle \leq \langle g_1g_2g_3, g_3, g_4 \rangle \leq G$
$N_3 = \langle g_1, g_3, g_4 \rangle$	$\{1\} \leq \langle g_4 \rangle \leq \langle g_1, g_3, g_4 \rangle \leq G$
$N_4 = \langle g_2, g_3, g_4 \rangle$	$\{1\} \leq G$
$N_5 = \langle g_3, g_4 \rangle$	$\{1\} \leq G$
$N_6 = \langle g_4 \rangle$	$\{1\} \leq G$
$N_7 = \{I\}$	$\{1\} \leq G$

By Lemma 2.2 we have that the generalised Wielandt series for $N_1 = G$ is the same as Wielandt series for G. Hence for $i \in \{1, 2, .., 7\}$ we have that $wl_{N_i}(G) \leq wl(G) = 3$.

3.1 Supersoluble groups

The aim of this section is to find a relationship between the generalised Wielandt length of a supersoluble group and certain invariants of its Sylow subgroups, see also [3].

3.2. Theorem: *Let G be a supersoluble group with $M = O_{p'}(G), N \trianglelefteq G$. If A is a Sylow p-subgroup of G such that $N \subseteq A$. Then*

$$w_{MN}(AM) \cap Fit(G) = O_p(w_{MN}G).$$

Proof: "$\subseteq$" Let G satisfies the hypothesis of the theorem. Then we have that

$$(w_{MN}(AM) \cap Fit(G))M/M \subseteq w_{MN}(AM)M/M.$$

Since G/M is a supersoluble group and by [29, Theorem 5.4.8] $\mathrm{Fit}(G/M)$ is a p-group. Thus G/M has a normal Sylow p-subgroup. Since AM/M is a Sylow p-subgroup of G/M, therefore $AM/M \trianglelefteq G/M$. Now by Theorem 2.21(i), Corollary 2.18 and Theorem 2.23(ii) we have that

$$w_{MN}(AM)M/M \subseteq w_{MN/M}(AM/M) \subseteq w_{MN/M}(G/M) = w^p_{MN}(G)/M.$$

Therefore it follows that

$$w_{MN}(AM) \cap Fit(G) \subseteq w^p_{MN}(G).$$

Since $w_{MN}(AM) \cap Fit(G)$ is a subnormal p-subgroup of G. Therefore for all primes $p \neq q$ we have that

$$w_{MN}(AM) \cap Fit(G) \subseteq O_{q'}(G).$$

But by Theorem 2.21(iv), $O_{q'}(G) \subseteq w^q_{MN}(G)$. Therefore by using Theorem 2.22, we will get

$$w_{MN}(AM) \cap Fit(G) \subseteq w_{MN}(G).$$

Again since $w_{MN}(AM) \cap Fit(G)$ is a p-group, therefore $w_{MN}(AM) \cap Fit(G) \subseteq O_p(w_{MN}(G))$.
"$\supseteq$" It follows by definition that $O_p(w_{MN}(G)) \subseteq w_{MN}(AM)$. Further $O_p(w_{MN}(G))$ is a nilpotent and characteristic subgroup of $w_{MN}(G)$. Therefore $O_p(w_{MN}(G)) \trianglelefteq G$ and $O_p(w_{MN}(G))$ is contained in $\mathrm{Fit}(G)$. Therefore we have that

$$O_p(w_{MN}(G)) \subseteq w_{MN}(AM) \cap Fit(G).$$

Hence the result follows. •

For a nilpotent group G we denote with $cl(G)$ its nilpotency class. Recall that this is the length of the lower central series $G = \gamma_1(G) \geq \gamma_2(G) \geq \ldots$. The following lemma is a preparation for our later results.

3.3. Lemma: *Let G be a finite supersoluble group and $N \trianglelefteq G$. Let A be a non-abelian Sylow p-subgroup of G with $N \subseteq A$. Then each Sylow p-subgroup of $G/w_N(G)$ has nilpotency class at most $cl(A)-1$.*

Proof: Denote $cl(A)=n$. Then

$$\gamma_n(A) \subseteq Z(A) \subseteq w(A) \subseteq w_N(A).$$

By Corollary 2.18, we have that $w_N(A) \subseteq w_N(G)$. Hence $\gamma_n(A) \subseteq w_N(G)$ follows. Since $Aw_N(G)/w_N(G)$ is a Sylow p-subgroup of $G/w_N(G)$ and

$$\gamma_n(Aw_N(G)/w_N(G)) = \gamma_n(A)w_N(G)/w_N(G) = \{1\}.$$

Therefore $Aw_N(G)/w_N(G)$ has nilpotency class at most $n-1$. •

3.4. Theorem: *Let G be a supersoluble group with $N \trianglelefteq G$. Suppose that all Sylow p-subgroups of G for odd p are abelian and the Sylow 2-subgroups have class at most two. Then $wl_N(G) \leq 2$.*

Proof: Let us suppose that, for some prime p dividing $|G|$, $O_{p'}(G)=1$. In this case by [29, Theorem 5.4.8] $O_p(G)$ is a normal Sylow p-subgroup of G. First we consider the case $p \neq 2$. Then by hypothesis $O_p(G)$ is abelian and hence by Lemma 2.17 $O_p(G) \subseteq w_N(G)$. Since $O_p(G) = Fit(G)$ therefore $G/O_p(G)$ is abelian by [29, Theorem 5.4.10]. Hence it follows that $G/w_N(G)$ is abelian and

$$w_{Nw_N(G)/w_N(G)}(G/w_N(G)) = G/w_N(G).$$

But by definition

$$w_{Nw_N(G)/w_N(G)}(G/w_N(G)) = w_{N,2}(G)/w_N(G).$$

Hence $w_{N,2}(G)=G$ and therefore $wl_N(G) \leq 2$.
Now consider the case when $p=2$. In this case by [29, Theorem 5.4.8] $O_p(G)=G$ and thus G has nilpotency class at most two. As $Z(G) \subseteq w_N(G)$, it follows that $G/w_N(G)$ is abelian. Thus under the assumption $O_{p'}(G)=1$ we obtain that $wl_N(G) \leq 2$.
Now consider the general case and for p dividing $|G|$, let $H = G/O_{p'}(G)$ and $K = NO_{p'}(G)/O_{p'}(G) \trianglelefteq H$. Note that $O_{p'}(H)=1$. Further by using Theorem 2.21(iii) and Theorem 2.23(ii) the local generalised Wielandt subgroup of H is given by

$$\begin{aligned} w_K^p(H) &= w^p_{NO_{p'}(G)/O_{p'}(G)}(G/O_{p'}(G)) \\ &= w^p_{NO_{p'}(G)}(G)/O_{p'}(G) \\ &= w_{NO_{p'}(G)/O_{p'}(G)}(G/O_{p'}(G)) \\ &= w_K(H). \end{aligned}$$

By the fact that under the assumption $O_{p'}(G) = 1$, $G/w_N(G)$ is abelian. It follows that $H/w_K(H)$ is abelian and hence $H/w^p_K(H)$ is abelian. Now by using Theorem 2.21(iii) it follows that

$$\begin{aligned} H/w^p_K(H) &= (G/O_{p'}(G))/(w^p_{NO_{p'}(G)/O_{p'}(G)}(G/O_{p'}(G))) \\ &= (G/O_{p'}(G))/(w^p_{NO_{p'}(G)}(G)/O_{p'}(G)) \\ &\cong G/w^p_{NO_{p'}(G)}(G). \end{aligned}$$

Thus $G/w^p_{NO_{p'}(G)}(G)$ is abelian, hence $G' \subseteq w^p_{NO_{p'}(G)}(G)$ for all primes dividing $|G|$. Therefore $G' \subseteq \bigcap_{p\in\pi} w^p_{NO_{p'}(G)}(G)$ and hence by Theorem 2.22 $G' \subseteq w_{NO_{p'}(G)}(G)$. Thus $G/w_{NO_{p'}(G)}(G)$ is abelian. Hence the result follows. •

Now by using Theorem 3.4 we obtain the following theorem, see also Theorem 1.16(ii).

3.5. Theorem: *Let G be a supersoluble group and let $N \trianglelefteq G$. Let n be the maximum of the nilpotency classes of those Sylow p-subgroups H of G with $N \leq H$. Then $wl_N(G) \leq n+1$.*

Proof: We will proceed by induction on n. If $n = 1$, then by Theorem 3.4 G has generalised Wielandt length $n+1$. Thus suppose that $n > 1$. Then all Sylow subgroups of G containing N have nilpotency class at most n. By Lemma 3.3, it follows that all Sylow subgroups of $G/w_N(G)$ have nilpotency class at most $n-1$. Thus by induction it follows that $G/w_N(G)$ has generalised Wielandt length at most n with respect to $Nw_N(G)/w_N(G)$. Thus $w_{N,n+1}(G) = G$ and the result follows. •

3.2 A characterisation

The aim of this section is to exhibit a characterisation of the higher order generalised Wielandt subgroups. We first recall the following lemma.

3.6. Lemma: [11, Lemma 2.1] *Let G be a group with $H, N \trianglelefteq G$ and $K \leq G$ such that $H \leq K \leq N \leq G$. Then*

(i) $N_{G/H}(K/H) = N_G(K)/H$.

(ii) *$K \lhd\lhd G$ if and only if $K/H \lhd\lhd G/H$.*

3.7. Theorem: *Let G be a finite group and $N \trianglelefteq G$. Then for $i \geq 0$,*

$$w_{N,i+1}(G) = \cap\{N_G(K) \mid w_{N,i}(G) \leq K \leq Nw_{N,i}(G), K \lhd\lhd G\}.$$

Proof: The case for $i = 0$, follows directly from the definition of $w_{N,i}(G)$. Now consider the case for $i \geq 1$ and write $L = w_{N,i}(G)$ to shorten notation. Then using the definition of $w_{N,i}(G)$ and Lemma 3.6 it follows that

$$\begin{aligned} & w_{N,i+1}(G)/L, \\ = \; & w_{NL/L}(G/L) \\ = \; & \bigcap \{N_{G/L}(K/L) \mid K/L \leq NL/L \text{ and } K/L \lll G/L\}, \\ = \; & \bigcap \{N_G(K)/L \mid K/L \leq NL/L \text{ and } K/L \lll G/L\}, \\ = \; & \bigcap \{N_G(K)/L \mid L \leq K \leq NL \text{ and } K \lll G\}, \\ = \; & \bigcap \{N_G(K) \mid L \leq K \leq NL \text{ and } K \lll G\}/L. \end{aligned}$$

Hence the theorem holds for $i \geq 1$.

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3.3 The generalised Wielandt length and factor groups

In the following results we establish the relation between the generalised Wielandt length of a finite group and the generalised Wielandt length of a certain factor group.

3.8. Proposition: *If G is a finite group and let n be fixed positive integer, then*

$$w_{N,n-m}(G/w_{N,m}(G)) = w_{N,n}(G)/w_{N,m}(G)$$

for all integers m with $0 \leq m \leq n$.

Proof: Let $n - m = 0$, then

$$w_{N,0}(G/w_{N,m}(G)) = w_{N,m}(G)/w_{N,m}(G)$$

and hence $w_{N,m}(G) = w_{N,m}(G)$.
Suppose the result is true for $k = n - m \geqslant 0$, that is

$$w_{N,k}(G/w_{N,m}(G)) = w_{N,m+k}(G)/w_{N,m}(G). \tag{3.1}$$

Now consider the case for $n - m = k + 1$ and write $L = w_{N,m}(G)$ to shorten notation. Then using Theorem 3.7 and Lemma 3.6(i) and (3.1) it follows that

$$\begin{aligned}
& w_{N,k+1}(G/L) \\
= \; & \bigcap\{N_{G/L}(K/L) \mid w_{N,k}(G/L) \leq K/L \leq Nw_{N,k}(G/L) \text{ and } K/L \lll G/L\}, \\
= \; & \bigcap\{N_G(K)/L \mid w_{N,m+k}(G)/L \leq K/L \leq Nw_{N,m+k}(G)/L \text{ and } K/L \lll G/L\}, \\
= \; & \bigcap\{N_G(K) \mid w_{N,m+k}(G) \leq K \leq Nw_{N,m+k}(G) \text{ and } K \lll G\}/L, \\
= \; & w_{N,m+k+1}(G)/L.
\end{aligned}$$

Hence the result follows. •

3.9. Theorem: *Let G be a finite group, let $N \trianglelefteq G$ and let $m, n \geq 0$ be integers. If the generalised Wielandt length of $G/w_{N,m}(G)$ with respect to $Nw_{N,m}(G)/w_{N,m}(G)$ is n, then $wl_N(G) = m+n$.*

Proof: We will prove this theorem by induction on m, since $G/w_{N,m}(G)$ has generalised Wielandt length n, therefore by definition we have that

$$w_{N,n}(G/w_{N,m}(G)) = G/w_{N,m}(G).$$

For $m = 0$ we have that

$$w_{N,n}(G/w_{N,0}(G)) = G/w_{N,0}(G)$$

which implies that $w_{N,n}(G) = G$. Hence the theorem holds for $m = 0$.
Now let $m > 0$. As $G/w_{N,m}(G)$ has generalised Wielandt length n, then by Proposition 3.8, we have that

$$w_{N,m+n}(G)/w_{N,m}(G) = w_{N,n}(G/w_{N,m}(G)) = G/w_{N,m}(G),$$

and hence $w_{N,m+n}(G) = G$.
Thus the generalised Wielandt length of G is atmost $m + n$. We claim that the generalised Wielandt length of G is exactly $m + n$. If $w_{N,m+n-1}(G) = G$, then by Proposition 3.8 we have that

$$G/w_{N,m}(G) = w_{N,m+n-1}(G)/w_{N,m}(G) = w_{N,n-1}(G/w_{N,m}(G)).$$

Which shows that $G/w_{N,m}(G)$ has generalised Wielandt length $n - 1$, a contradiction to the hypothesis. Hence $w_{N,m+n-1}(G) \neq G$ and this implies that G must have generalised Wielandt length $m + n$. •

3.4 The groups with generalised Wielandt length one

The aim of this section is to exhibit the properties of a group G with $wl_N(G) = 1$.

3.10. Remark:

(i) If G is a T-group, then $wl_N(G) = 1$.
(ii) If $M, N \trianglelefteq G$ with $M \subseteq N$ and $wl_N(G) = 1$, then $wl_M(G) = 1$.

The converse of Remark 3.10(i) is not true.

3.11. Example: Let $G = D_8$ and $N = C_4$, then $wl_N(G) = 1$ but G is not a T-group.

3.12. Proposition: *If $wl_N(G) = 1$ and N is a minimal normal subgroup of G, then N is simple.*

Proof: Let us supposed that N is not simple. Let N_1 be a proper normal subgroup of N. Therefore N_1 is subnormal in G and contained in N, hence $N_1 \trianglelefteq G$. This is a contradiction to the hypothesis. Hence the result follows. •

3.13. Theorem: *Let G be a group and $N, M \trianglelefteq G$ with $M \leq N$. If $wl_N(G) = 1$, then $wl_{N/M}(G/M) = 1$.*

Proof: Let $wl_N(G) = 1$ and S/M be a subnormal subgroup of G/M contained in N/M. Let x be an element of S/M and y be an element of G/M, then $x = sM$ and $y = gM$ for some $x \in S$ and $g \in G$. Now consider

$$yxy^{-1} = (gM)(sM)(gM)^{-1} = (gM)(sM)(g^{-1}M) = (gsg^{-1})(M) \in S/M.$$

Thus $S/M \trianglelefteq G/M$ and hence $wl_{N/M}(G/M) = 1$. •

3.14. Theorem: *Let G be a group. If $wl_{Fit(G)}(G) = 1$, then $C_G(\gamma_2(G)) = Fit(G)$ and Fit(G) is a Dedekind group.*

Proof: Let $C = C_G(\gamma_2(G))$. Then $[C, \gamma_2(C)] \leq [C, \gamma_2(G)] = \{1\}$. Thus C is nilpotent and $C \leq Fit(G)$. Let M be a normal nilpotent subgroup of G and let $x \in M$. Then $\langle x \rangle \lhd\lhd M \leq Fit(G) \trianglelefteq G$, but $wl_{Fit(G)}(G) = 1$, therefore $\langle x \rangle \trianglelefteq G$. Further $G/C_G(x) \cong Aut\langle x \rangle$ and by [29, Theorem 1.5.5 (ii)] $Aut\langle x \rangle$ is abelian. Hence $G' \leq C_G(x)$ and x belongs to C. It follows that $M \leq C$ and thus $C = Fit(G)$. Furthermore since $wl_{Fit(G)}(G) = 1$ we have that $Fit(G)$ is Dedekind. •

3.15. Theorem: *Every soluble group G with $wl_{Fit(G)}(G) = 1$ is metabelian.*

Proof: Let G be a soluble group. Suppose that there is an integer $n > 2$ such that $G^{(n)} = \{1\}$ but $G^{(n-1)} \neq \{1\}$. Let $H = G^{(n-3)}$, then H is soluble and H'' is a non-trivial abelian subgroup. Let $wl_{Fit(H)}(H) = 1$, then by Theorem 3.14 we have that $H'' \subseteq C_H(H')$. Therefore $[\gamma_2(H'), H'] = \{I\}$ and hence H' is nilpotent. Thus $H' \leq Fit(H) = C_H(H')$ and therefore $H'' = \{I\}$, contradiction. Hence the result follows.

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Chapter 4

The relative Wielandt subgroup

The aim of this chapter is to investigate properties of the relative Wielandt subgroup.

4.1 Structural properties

In this section we exhibit some structural properties of the relative Wielandt subgroup which explain the notion of the relative Wielandt subgroup in a better way.
In the following remark we exhibit the conditions under which the relative Wielandt subgroup is the full group.

4.1. Remark: Let G be a group, then $w_c(G) = G$ if G is one of the following.

(i) T_c-group.
(ii) Dedekind group.
(iii) Finite simple group.
(iv) Finite group with cyclic Sylow subgroups.

In the following theorem we investigate a class of groups in which the relative Wielandt subgroup and the Wielandt subgroup coincide, see also Theorem 1.23.

4.2. Theorem: *If G is a quasisimple group with cyclic center, then $w_c(G) = w(G)$.*

Proof: "$\supseteq$" It follows from the definition of $w_c(G)$ that $w(G) \leq w_c(G)$ always holds.
"$\subseteq$" Let G be a quasisimple group, then $G' = G$ and $G/Z(G)$ is simple. Let S be a subnormal subgroup G, then either $S \leq Z(G)$ or $S = G$. Now if $S \leq Z(G)$, then S is cyclic, thus S will be normalised by $w_c(G)$ and if $S = G$, then $w(G) = G$ and $S = \{1\}$ is the only cyclic subnormal subgroup of G therefore $w_c(G) = G$. Thus in both cases $w_c(G) \leq w(G)$. Hence the result follows. •

The relative Wielandt subgroup also has finite index in G.

4.3. Lemma: *Let G be a group. Then*

(i) *$w_c(G)$ contains every simple non-abelian subnormal subgroup of G.*

(ii) *$w_c(G)$ contains every minimal normal subgroup N of G such that N satisfies Min-sn.*

(iii) *If G satisfies Min-sn, then $w_c(G)$ has finite index in G.*

Proof: (i) Let S be a simple non-abelian subnormal subgroup of G, then by [34, Theorem 1(a)], $S \leq w(G)$. Hence the result follows from $w(G) \leq w_c(G)$.
(ii) Let N satisfies Min-sn, then by [34, Theorem 1(b)] we have that $N \leq w(G)$. Hence the result follows from $w(G) \leq w_c(G)$.
(iii) By [29, Theorem 13.3.8] we obtain that $[G : w(G)]$ is finite. Thus the result follows from $w(G) \leq w_c(G)$. •

Since $Z(G) \leq N(G) \leq w_c(G)$ and every element of $N(G)$ induces a power automorphism($Paut(G)$) of G. Further $G/Z(G) \cong Inn(G)$. Using these relations, we prove the following.

4.4. Theorem: *For any finite group G, $w_c(G)/Z(G) \cong Inn(G) \cap Paut(G)$.*

Proof: Let $\tau : w_c(G) \to Inn(G) \cap Paut(G)$ be a mapping defined by $\tau(g) = g^\tau$ and $g^\tau : G \to G$ by $g^\tau(x) = x^g$. Now consider

$$\begin{aligned}
(g_1g_2)^\tau(x) &= (x)^{(g_1g_2)} \\
&= g_1g_2(x)(g_1g_2)^{-1} \\
&= (g_1g_2)(x)(g_2^{-1}g_1^{-1}) \\
&= g_1(g_2xg_2^{-1})g_1^{-1} \\
&= g_1(x^{g_2})g_1^{-1} \\
&= g_1(g_2^\tau(x))g_1^{-1} \\
&= g_1^\tau g_2^\tau(x).
\end{aligned}$$

Therefore $\tau(g_1g_2) = (g_1g_2)^\tau = g_1^\tau g_2^\tau = \tau(g_1)\tau(g_2)$. Thus τ is a homomorphism. To show that τ is bijective. Let us suppose that $g_1 \neq g_2$, then $x^{g_1} \neq x^{g_2}$ which implies that $g_1^\tau \neq g_2^\tau$. Further for each $g^\tau \in Inn(G) \cap Paut(G)$ there exist g belongs to $w_c(G)$ such that $g^\tau(x) = x^g$. Hence τ is bijective. To complete the proof we need to show that $ker(\tau) = Z(G)$. For this let $g \in ker(\tau)$, then $g^\tau(x) = I_{aut}(x) = x$ which implies that $xg = gx$ for all $x \in G$, hence $ker(\tau) \leq Z(G)$. In the other direction let $g \in Z(G)$, then for all $x \in G$ we have that $x^g = x = I_{aut}(x) = g^\tau(x)$, which implies that $g \in ker(\tau)$ and $Z(G) \leq ker(\tau)$. Hence $ker(\tau) = Z(G)$ and $w_c(G)/Z(G) \cong Inn(G) \cap Paut(G)$ as required. •

The properties that are describe in the following theorem do not hold in the Wielandt subgroup, see also Theorem 1.24.

4.5. Theorem: *Let G and H be a groups. Then*
(i) $w_c(G \times H) = w_c(G) \times w_c(H)$.
(ii) *If $G = D_\infty$, then $w_c(G) = G$.*

Proof: (i) "$\subseteq$" Let $g = (g_1, h_1)$ be an element of $w_c(G \times H)$ and $S = (A \times B)$ be a cyclic subnormal subgroup of $G \times H$, where A is a cyclic subnormal subgroup of G and B is a cyclic subnormal subgroup of H. Now consider

$$\begin{aligned} S &= (S)^g \\ &= (A \times B)^g \\ &= g(A \times B)g^{-1} \\ &= (g_1, h_1)(A \times B)(g_1^{-1}, h_1^{-1}) \\ &= (A^{g_1} \times B^{h_1}). \end{aligned}$$

Thus we have that $A^{g_1} = A$, $B^{h_1} = B$ and hence $g = (g_1, h_1) \in w_c(G) \times w_c(H)$.
"$\supseteq$" Let (g, h) be an element of $w_c(G) \times w_c(H)$ therefore $A^g = A$ for all cyclic subnormal subgroups A of G and $B^h = B$ for all cyclic subnormal subgroups B of H. Let us suppose that $A \times B$ be a cyclic subnormal subgroup of $G \times H$. Then we have that $(A \times B)^{(g,h)} = A^g \times B^h = A \times B$, which implies that $(g, h) \in w_c(G \times H)$. Hence the result follows.
(ii) Let $G = D_\infty$, then by [31, Corollary 8] G is a T_c-group. Thus by Remark 4.1(i) we have that $w_c(G) = G$. •

Next, in the following results we exhibit some properties of the relative Wielandt subgroup for polycyclic groups, see also Theorem 1.25.

4.6. Lemma: *Let N be a normal subgroup of a polycyclic group G. If N centralises $Fit(G)$, then $N \leq Z(Fit(G))$.*

Proof: To show that $N \leq Fit(G)$. Let us suppose that $N \nleq Fit(G)$, so this implies that there exists $A \lhd G$ with $A/Fit(G) \neq \{1\}$ abelian and $A/Fit(G) \leq NFit(G)/Fit(G)$ which implies that $Fit(G) \lneq A \leq NFit(G)$ and hence A is a nilpotent normal subgroup of G, contradiction. Hence $N \leq Fit(G)$ and by [29, Theorem 5.4.4(ii)] we have that $N \leq Z(Fit(G))$. •

4.7. Theorem: *Let G be a polycyclic group. Then*
(i) $w_c(G)Z(Fit(G))/Z(Fit(G)) \cong w_c(G)/w_c(G) \cap Z(Fit(G))$ *is finite abelian.*
(ii) $w_c(G)$ *is supersoluble and metabelian.*

Proof: (i) Since $Fit(G)$ is nilpotent therefore $w_c(G)$ normalises each cyclic subgroup of $Fit(G)$. Thus $w_c(G)$ acts as group of power automorphisms on $Fit(G)$. Therefore $w_c(G)/w_c(G) \cap Fit(G)$ is abelian. Further let φ be an Euler's totient function and $\sigma = 2.lcm\{\varphi(\mid g \mid) : g \in Fit(G), \mid g \mid < \infty\}$. Let $h \in w_c(G)$ and we aim to show that h^σ centralises $Fit(G)$. For this let g belongs to $Fit(G)$. Then we have that $\mid g \mid = \infty$ or $\mid g \mid < \infty$. If $\mid g \mid = \infty$, then $g^h = g^{\pm 1}$ and hence $g^{h^2} = g$. If $\mid g \mid < \infty$, then $g^{h^{\rho(\mid g \mid)}} = g$. Thus in both the cases h^σ centralises $Fit(G)$. Now let $\mu = < h^\sigma : h \in w_c(G) > \lhd G$, then by Lemma 4.6 we have that $\mu \leqslant Z(Fit(G))$. Hence $w_c(G)/w_c(G) \cap Z(Fit(G))$ is finite.
(ii) It follows from (i) that $w_c(G)$ is finitely generated. Therefore by [29, Theorem 5.4.6(ii)] $w_c(G)$ is supersoluble. Further since $w_c(G)$ is a soluble T_c-group therefore by [31, Theorem 3] we have that $w_c(G)$ is metabelian. •

4.8. Lemma: *Let G be a polycyclic group. If G is torsion free, then $w_c(G) \cap Fit(G) = w_c(G) \cap Z(Fit(G))$*

Proof: Let G be a torsion free group, then $Fit(G)$ is torsion free and by [29, Theorem 5.2.19] $Fit(G)/Z(Fit(G))$ is torsion free group. Thus $w_c(G) \cap Fit(G) = Z(Fit(G)) = w_c(G) \cap Z(Fit(G))$.
•

The following corollary is an immediate consequence of Lemma 4.8.

4.9. Corollary: *Let G be a polycyclic group. If G is torsion free nilpotent, then $w_c(G) = Z(G)$.*

4.2 The local relative Wielandt subgroup

The aim of this section to exhibit the relation between the relative Wielandt subgroup and its local counterpart.

4.10. Remark: Let G be a group, then
(i) $w(G) \leq w_c(G) \leq w_c^p(G)$.
(ii) $w(G) \leq w^p(G) \leq w_c^p(G)$.

The following theorem gives a relation between $w_c(G)$ and $w_c^p(G)$.

4.11. Theorem: *Let G be a finite group. If π is the set of all primes dividing $|G|$, then*

$$w_c(G) = \bigcap_{p\in\pi} w_c^p(G).$$

Proof: "$\subseteq$" It follows directly from the definition that $w_c(G) \subseteq w_c^p(G)$ for every prime p.
"$\supseteq$" Let S be a cyclic subnormal subgroup of G, then $S = p_1^{s_1}...p_\mu^{s_\mu}$ (prime factors of S), this implies that $S = C_{p_1^{s_1}}...C_{p_\mu^{s_\mu}}$, where each $C_{p_i^{s_i}} \lhd\lhd G$ and p_i'-perfect. Since $w_c(G)$ normalises S, therefore it follows that each $C_{p_i^{s_i}}$ is normalised by $w_c(G)$. Hence the result follows. •

4.12. Theorem: *Let G and H be finite groups and $N \lhd G$, then*
(i) *$w_c(G)N/N \leq w_c(G/N)$ and $w_c^p(G)N/N \leq w_c^p(G/N)$.*
(ii) *$w_c(G) \cap N \leq w_c(N)$ and $w_c^p(G) \cap N \leq w_c^p(N)$.*
(iii) *If N is the unique maximal normal subgroup of G. Then*
$w_c(G) \cap N = w_c(N)$ and $w_c^p(G) \cap N = w_c^p(N)$.
(iv) *$w_c^p(G \times H) = w_c^p(G) \times w_c^p(H)$.*

Proof: Since the proof in all cases (i) to (iii) for relative Wielandt subgroup and its local version are identical. We prove (i) to (iii) for the relative Wielandt subgroup only.
(i) Let $L = \cap_{U\lhd\lhd G, N\leq U, U \text{is cyclic}} N_G(U)$. Then $w_c(G) \leq L$ and $N \leq L$ and thus $w_c(G)N \leq L$. Further, $L/N = w_c(G/N)$, since the cyclic subnormal subgroups of G/N correspond one-to-one to the cyclic subnormal subgroups of G containing N and this correspondence extends to their normalisers.
(ii) Let $L = \cap_{U\lhd\lhd G, U\leq N, U \text{is cyclic}} N_G(U)$ and $R = \cap_{U\lhd\lhd G, U\not\leq N, U \text{is cyclic}} N_G(U)$. Then $w_c(G) = L \cap R$ by construction. Further, $w_c(N) = L \cap N$, since $N_N(U) = N_G(U) \cap N$ for every $U \leq G$. Thus $w_c(G) \cap N = L \cap R \cap N = w_c(N) \cap R \leq w_c(N)$.
(iii) Every proper cyclic subnormal subgroup of G is contained in N, thus

$$w_c(G) = \bigcap_{\substack{U\lhd\lhd G \\ U \text{is cyclic}}} N_G(U) = \bigcap_{\substack{U\lhd\lhd N \\ U \text{is cyclic}}} N_G(U)$$

and

$$w_c(G) \cap N = N \cap \bigcap_{\substack{U\lhd\lhd N \\ U \text{is cyclic}}} N_G(U) = \bigcap_{\substack{U\lhd\lhd N \\ U \text{is cyclic}}} N_N(U) = w_c(N).$$

(iv) The proof is similar to the proof of Theorem 4.5(ii) just replace cyclic subnormal subgroup by p'-perfect cyclic subnormal subgroup.

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Chapter 5

Algorithms

The aim in this chapter is to develop methods to compute the Norm and the generalised Wielandt subgroup in certain types of groups. We refer to [21] to background in algorithmic group theory and methods to compute with finite groups. In [13] various methods to compute with polycyclic groups are exhibited. All the methods described below are implemented in the computer algebra system GAP [17] using some of its packages, in particular, the Package "Polycyclic"[16] will be used for computations with infinite polycyclic groups.

5.1 Computing the norm

Our first aim is to develop a method to compute the Norm of a finite group or a (possibly infinite) polycyclic group. First note that an infinite polycyclic group G always contains an element of infinite order; thus $N(G) = Z(G)$ follows and $Z(G)$ can be computed as described in [15]. Hence it remains to consider finite groups G. In this case we use that $N(G)$ can also be described as the normaliser of all cyclic subgroups of G. Thus

$$N(G) = \bigcap_{g \in G} N_G(\langle g \rangle).$$

Further, note that if $g^x = h$ in G, then $N_G(\langle h \rangle) = N_G(\langle g \rangle)^x$. Recall that the core of a subgroup U in a group G is the intersection of all G-conjugates of U. Thus if $g_1, \ldots, g_l$ is a complete set of conjugacy class representatives of elements in G, then

$$N(G) = \bigcap_{i=1}^{l} Core_G(N_G(\langle g_i \rangle)).$$

This induces the following algorithm to compute $N(G)$. Note that the second center and the conjugacy classes of a finite group G can be computed by methods described in [21] as well as normalisers and cores of subgroups.

Algorithm Norm
Input: a group G that is polycyclic or finite.
Output: the Norm of G.

(1) Compute the center $Z(G)$.
(2) If $\mathrm{Size}(Z(G)) = 1$ or $|G| = \infty$, then return $Z(G)$.
(3) Compute the second center $Z_2(G)$ and initialise $N = Z_2(G)$.
(4) Compute the conjugacy classes of elements of G.
(5) For each class g^G do
  (i) Replace N by $N_N(\langle g \rangle)$.
  (ii) Replace N by its core $Core_G(N)$.
  (iii) If $N = Z(G)$ then return N.
(6) Return N.

This algorithm allows to determine the Norm for various types of groups readily as we exhibit in the following example.

5.1. Example:
(i) $N(S_n) = Z(S_n) = \{1\}$ for $n \geq 3$.
(ii) $N(D_{2^n}) = Z(D_{2^n})$ for $n \geq 1$.
(iii) $N(Q_{2^n}) = Z_2(Q_{2^n})$ for $n \geq 4$.

5.2 Computing the Wielandt subgroup and its variations

First note that for a group G the Wielandt subgroup $w(G)$ can also be described as $w_G(G)$ and thus computing the Wielandt subgroup of G is a special case of computing the generalised Wielandt subgroup.
The computation of the generalised Wielandt subgroup of a group G and a normal subgroup N is significantly more difficult than computing the norm. If one follows the definition of the generalised Wielandt subgroup, then one needs to determine the lattice of G-subnormal subgroups of N. If N is finite, then this is possible (see [21], Chapter 10), but usually rather time-consuming. If N is infinite, then it is not obvious at all how to do this. We exhibit the following algorithm for the case that N is finite.

Algorithm GeneralisedWielandtSubgroup
Input: a group G and a finite normal subgroup N of G.
Output: the generalised Wielandt subgroup $w_N(G)$.

(1) If G is nilpotent and $N = G$, then return $N(G)$.
(2) Initialise $W = G$.
(3) Compute the G-conjugacy classes of G-subnormal subgroups of N.
(4) For each conjugacy class U^G do
    (i) Replace W by $N_W(U)$.
    (ii) Replace W by its core $Core_G(W)$.
    (iii) If $W = Z(G)$ then return W.
(5) Return W.

The relative Wielandt subgroup for a finite group G can be computed with a variation of the "Algorithm GeneralisedWielandtSubgroup" by choosing $N = G$ and restricting to cyclic G-subnormal subgroups of G in step (3). We exhibit this for completeness as follows.

Algorithm RelativeWielandtSubgroup
Input: a finite group G.
Output: the relative Wielandt subgroup $w_c(G)$.

(1) If G is a nilpotent, then return $N(G)$.
(2) Initialise $W = G$.
(3) Compute the G-conjugacy classes of cyclic G-subnormal subgroups of G.
(4) For each conjugacy class U^G do
    (i) Replace W by $N_W(U)$.
    (ii) Replace W by its core $Core_G(W)$.
    (iii) If $W = Z(G)$ then return W.
(5) Return W.

Note that the relative Wielandt subgroup is significantly more practical than generalised Wielandt subgroup, since it only requires the computation of conjugacy classes of elements instead of conjugacy classes of subgroups.

There is currently no algorithm available to compute the Wielandt subgroup of an infinite group. The perhaps easiest case of infinite groups are the infinite polycyclic groups. We consider this case here briefly.

5.2. Lemma: *Let G be a polycyclic group. Then*

(i) $Z(G) \leq w(G) \cap Fit(G) \leq N(Fit(G)) \leq Z_2(Fit(G))$.

(ii) $w(G)/(w(G) \cap Fit(G))$ *is finite abelian and acts as power automorphisms on* $Fit(G)$.

(iii) $w(G)$ *is supersoluble and metabelian.*

Proof: (i) Follows from Theorem 2.21(ii) using that $w(Fit(G)) = N(Fit(G))$ since $Fit(G)$ is nilpotent.
(ii) Let $\rho : G \to Aut(Fit(G))$ be a homomorphism induced by conjugation of G on $Fit(G)$. Since every subgroup of $Fit(G)$ is subnormal in G and hence normalized by $w(G)$. Thus $w(G)$ acts as group of power automorphisms on $Fit(G)$. Hence $\rho(w(G)) \cong w(G)/w(G) \cap Fit(G)$ is abelian. Further $\rho(w(G)) \cong w(G)/w(G) \cap Fit(G)$ is finite by the arguments of Theorem 4.7(i) for $w(G)$. Hence the result follows.
(iii) It follows from (ii) that $w(G)$ is finitely generated. Therefore by [29, Theorem 5.4.6(ii)] $w(G)$ is supersoluble. Further since $w(G)$ is a soluble T-group therefore by [29, Theorem 13.4.2] we have that $w(G)$ is metabelian. •

Lemma 5.2 allows to "approximate" the Wielandt subgroup of a polycyclic group to some extend, as $Z(G)$ and $N(Fit(G))$ can be computed.

5.2.1 Computing the generalised Wielandt series

The computation of the generalised Wielandt series is a simple induction using the generalised Wielandt subgroup algorithm. We exhibit this in the following.

Algorithm GeneralisedWielandtSeries
Input: a group G and a finite normal subgroup N.
Output: the generalised Wielandt series of G wrt N.

(1) Create an empty list S. Add the trivial subgroup to S.
(2) Initialise W as $W = w_N(G)$.
(3) While W is not contained in S do
    (i) Add W to S.
    (ii) Replace W with the preimage of $w_{NW/N}(G/W)$ under $G \to G/W$.
(4) Return S.

Using this algorithm, we obtain the following.

5.3. Example:

(1) Let $G = D_{2^n}$. Then for $n \geq 4$, G has exactly three normal subgroups of order 2^{n-1}, one is isomorphic to $C_{2^{n-1}}$ and the other two are isomorphic to $D_{2^{n-1}}$. By Remark 2.1 for $N = C_{2^{n-1}}$ we have that $wl_N(G) = 1$. While for $N \cong D_{2^{n-1}}$ or $N = G$, $wl_N(G) = n - 1$. For the rest of the proper normal subgroups of G, $wl_N(G) = 1$. By Lemma 2.2 we have that $wl(G) = n - 1$. Since both of the lengths are equal, their exists an interesting relationship between the corresponding terms of both of the series.

   (i) Let $n - 1 = m$(say), then
   $w_m(G) = w_{N,m}(G)$ for $m = 1, 2, .., n - 3$.

   (ii) $w_m(G) \leq w_{N,m}(G)$ for $m = n - 2$.

(2) Let $G = Q_{2^n}$. Then for $n \geq 5$, G has exactly three normal subgroups of order 2^{n-1}, one is isomorphic to $C_{2^{n-1}}$ and the other two are isomorphic to $Q_{2^{n-1}}$. For $N = C_{2^{n-1}}$ we have that $wl_N(G) = 1$, while for $N \cong Q_{2^{n-1}}$ or $N = G$, $wl_N(G) = n - 2$. For the rest of the proper normal subgroups of G, $wl_N(G) = 1$. By Lemma 2.2 we have that $wl(G) = n - 2$. Since both of the lengths are equal, the relationship between the corresponding terms of both of the series is the following.

   (i) Let $n - 2 = m$(say), then
   $w_m(G) = w_{N,m}(G)$ for $m = 1, 2, .., n - 4$.

   (ii) $w_m(G) \leq w_{N,m}(G)$ for $m = n - 3$.

5.3 Checking for T-groups

We include a method to check if a given finite group G is a T-group.

Algorithm for T-groups
Input: a finite group G.
Output: true or false.

(1) Compute all normal subgroups of G.

(2) For each normal subgroup H of G do

   (i) Compute all normal subgroups of H.

   (ii) Check that each of them is normal in G.

   (iii) If one of them is not normal in G, then return false.

(3) Return true.

Chapter 6

Groups of order dividing p^6 with large Wielandt subgroup

A finite p-group G is nilpotent and hence the Wielandt subgroup of G coincides with the Norm of G. By Schenkman [32] it is known that $Z(G) \leq N(G) \leq Z_2(G)$ holds. Hence the Wielandt subgroup of G is in some sense maximal if $w(G) = Z_2(G)$ holds. The aim of this chapter is to exhibit the groups G of order dividing p^6 satisfying that $w(G) = Z_2(G)$. Note that the groups of order dividing p^6 are classified in [24] and we are using this classification as a basis. We also note that the central question of this chapter has been asked before as follows.

6.1. Question: (Guo; [19, Question 2.2])
What is known about the structure of the groups G with $N(G) = Z_2(G)$?

6.1 Structural results and conjecture

The following results gives a structural information of finite finite p-group with $w(G) = Z_2(G)$, see also Theorem 1.27.

6.2. Theorem: *Let G be a finite p-group with $w(G) = Z_2(G)$. Then $w(G)/Z(G) = w(G/Z(G)) = Z(G/Z(G))$.*

Proof: "$\subseteq$" By [8, Lemma 3.1(i)] it follows that $w(G)Z(G)/Z(G) \subseteq w(G/Z(G))$. Since $w(G) = Z_2(G)$, it follows that $w(G)/Z(G) \subseteq w(G/Z(G))$.
"$\supseteq$" Let $xZ(G)$ be an arbitrary element of $w(G/Z(G))$. Let S be a subnormal subgroup in G. Then x normalises $SZ(G)$ and hence x normalises S. Therefore $x \in w(G)$. •

The next theorem is a generalised form of [36, Theorem 3.3(3(ii))], see also Theorem 1.28.

6.3. Theorem: *Let G be a finite p-group with $Exp(G) = p$. Then $N(G) = w(G) = Z(G)$.*

Proof: Let G be a finite p-group and $g \in G$. Then by [29, Theorem 1.6.13] we have that $N_G(\langle g\rangle)/C_G(\langle g\rangle) \hookrightarrow Aut(\langle g\rangle)$. If $|g| = p$, then by [29, Theorem 1.5.5], $|Aut(\langle g\rangle)| = p-1$. Hence $N_G(\langle g\rangle) = C_G(\langle g\rangle)$. Further if $Exp(G) = p$, then $N(G) = \bigcap_{g\in G} N_G(\langle g\rangle) = \bigcap_{g\in G} C_G(\langle g\rangle) = Z(G)$. Hence the result follows. •

Further, we propose the following conjecture based on the experimental evidence of the Section 6.2.

6.4. Conjecture: Let G be a finite p-group with $w(G) = Z_2(G)$. Then $[Z_2(G) : Z(G)] = p$ for $p \geq 7$.

6.2 Enumeration results and conjectures

If G is abelian, then $Z(G) = w(G) = Z_2(G)$ holds. Hence in this case the Wielandt subgroup is maximal for trivial reasons and we exclude this case in the remainder of this section.
For $n \in \mathbb{N}$ let $g(n)$ denote the number of non-abelian groups of order n and let $f(n)$ denote the number of those non-abelian groups G of order n with $w(G) = Z_2(G)$ (in both cases groups are considered up to isomorphism).
We recall that $g(p^3) = 2$ for all primes p by Burnside [9, page 99-102] .

6.5. Theorem: *$f(2^3) = 1$ and $f(p^3) = 0$ for all primes $p > 2$.*

Proof: Let G be a non-abelian p-group of order p^3. Since all the non-abelian groups G of order p^3 are extra-special, therefore $Z_2(G) = G$. If p is an even prime, then D_8 and Q_8 are the only non-abelian groups of order 2^3. Since Q_8 is a T-group, therefore we have that $w(Q_8) = Q_8 = Z_2(Q_8)$. D_8 is a non T-group therefore, $w(D_8) \neq D_8 = Z_2(D_8)$. If p is an odd prime, then by Burnside [9] G is isomorphic to one of the following groups;
$G_1 \cong \langle a, b, c \mid a^p = b^p = c^p = 1, c^{-1}bc = ba, c^{-1}ac = a, ab = ba\rangle$
$G_2 \cong \langle a, b \mid a^{p^2} = b^p = 1, b^{-1}ab = a^{1+p}\rangle$.
If G is of type G_1, then $Exp(G) = p$ and by Theorem 6.3 we have that $w(G) = Z(G)$. If G is of type G_2, then $Exp(G) = p^2$ and $Z_2(G) = G \nleq (N_G(\langle a\rangle) \cap C_G(b)) = \langle b, aba^{-1}, a^3, a^2ba^{-1}\rangle$. Thus by [36, Theorem 3.3(3)(iii)] it follows that if G is of type G_2, then $Z_2(G) \neq w(G)$. Hence the result follows. •

We recall that $g(p^4) = 9$ if $p = 2$ and $g(p^4) = 10$ for all prime $p > 2$ by Burnside [9, page 99-102].

6.6. Theorem: *$f(2^4) = 2$ and $f(p^4) = 1$ for all primes $p > 2$.*

Proof: Let G be a non-abelian group of order p^4. For $p = 2$, $f(p^4)$ can be computed easily using the methods of Chapter 5 and the well known class of groups of order 16.
If p is an odd prime, then by Burnside [9, page 99-102] we have that G is isomorphic to one of the following groups.
$G_1 \cong \langle a, b \mid a^{p^3} = b^p = 1, ba = a^{1+p^2}b\rangle$.
$G_2 \cong \langle a, b, c \mid a^{p^2} = b^p = c^p = 1, cb = a^p bc, ab = ba, ac = ca\rangle$.
$G_3 \cong \langle a, b \mid a^{p^2} = b^{p^2} = 1, ba = a^{1+p}b\rangle$.
$G_4 \cong \langle a, b, c \mid a^{p^2} = b^p = c^p = 1, ca = a^{1+p}c, ba = ab, cb = bc\rangle$
$G_5 \cong \langle a, b, c \mid a^{p^2} = b^p = c^p = 1, ca = abc, ab = ba, bc = cb\rangle$.
$G_6 \cong \langle a, b, c, d \mid a^p = b^p = c^p = d^p = 1, dc = acd, bd = db, ad = da, bc = cb, ac = ca, ab = ba\rangle$.
$G_7 \cong \langle a, b, c \mid a^{p^2} = b^p = c^p = 1, ba = a^{1+p}b, ca = abc, cb = bc\rangle$.
If $p = 3$, $G_8 \cong \langle a, b, c \mid a^{p^2} = b^p = 1, c^p = a^p, ab = ba^{1+p}, ac = cab^{-1}, cb = bc\rangle$.
If $p > 3$, $G_8 \cong \langle a, b, c \mid a^{p^2} = b^p = c^p = 1, ba = a^{1+p}b, ca = a^{1+p}bc, cb = a^p bc\rangle$.
If $p = 3$, $G_9 \cong \langle a, b, c \mid a^{p^2} = b^p = 1, c^p = a^{-p}, ab = ba^{1+p}, ac = cab^{-1}, cb = bc\rangle$.
If $p > 3$, $G_9 \cong \langle a, b, c \mid a^{p^2} = b^p = c^p = 1, ba = a^{1+p}b, ca = a^{1+dp}bc, cb = a^{dp}bc, d \not\equiv 0, 1 (mod p)\rangle$.
If $p = 3$, $G_{10} \cong \langle a, b, c \mid a^{p^2} = b^p = c^p = 1, ab = ba, ac = cab, bc = ca^{-p}b\rangle$.
If $p > 3$, $G_{10} \cong \langle a, b, c, d \mid a^p = b^p = c^p = d^p = 1, dc = bcd, db = abd, ad = da, bc = cb, ac = ca, ab = ba\rangle$.
Further we note that all these groups are non Dedekind groups.
If G is a group of type G_1 to G_6, then G satisfies [26, Theorem 4.8(i)] and therefore $G/Z(G)$ is abelian. Thus $Z_2(G)/Z(G) = Z(G/Z(G)) = G/Z(G)$ and hence $Z_2(G) = G$. Further let us suppose that $w(G) = Z_2(G) = G$. Then it follows that G is a T-group and consequently Dedekind, thus we obtain a contradiction. Hence $w(G) \neq Z_2(G)$ for all groups of type G_1 to G_6.
If G is a group of type G_7 to G_9 and $p > 3$, then $|Z(G)| = p$ and $G/Z(G)$ is a non-abelian group of order p^3. Therefore by [26, Theorem 4.6(ii)] we have that $G/Z_2(G)$ is an abelian group and therefore $Z_3(G)/Z_2(G) = Z(G/Z_2(G)) = G/Z_2(G)$. Hence G is a regular p-group of maximal class. Further, $Exp(G) = p^2 > Exp(Z(G)) = p$ and order$(a) >$ order$(b) =$ order(c).
Now if G is of type G_7, then $Z_2(G) = \langle b, aba^{-1}\rangle \leq (N_G(\langle a\rangle) \cap C_G(b) \cap C_G(c)) = \langle b, aba^{-1}\rangle$. If G is of type G_8 or G_9, then $Z_2(G) = \langle b, aba^{-1}\rangle \not\leq (N_G(\langle a\rangle) \cap C_G(b) \cap C_G(c)) = \langle bab^{-1}a^{-1}\rangle$. Thus by [36, Theorem 3.3(3)(iii)] it follows that if G is of type G_7, then $Z_2(G) = w(G)$.
If G is a group of type G_{10} and $p > 3$, then we have that $Exp(G) = p$. Thus by Theorem 6.3 it follows that $w(G) = Z(G)$.
Finally let $p = 3$ and G be a group of type G_7 to G_{10}. Then $Z(G) = \langle a^3\rangle$ and $Z_2(G) = \langle b, a^3\rangle$. If G is of type G_7, G_9 or G_{10}, then $\langle a^2c\rangle \lll G$. Since $(a^2c)^b = b^{-1}a^2cb \notin \langle a^2c\rangle$, therefore $b \notin w(G)$

and $w(G) \neq Z_2(G)$. If G is of type G_8, then $\langle b \rangle$ is quasi normal in G. Thus by [19, Remark 2.1] it follows that $w(G) = Z_2(G)$. Hence the result follows. •

Using Theorem 6.5 and Theorem 6.6 and various computer experiments based on the methods of Chapter 5, we now give a summary on $g(p^n)$ and $f(p^n)$ for small n. The red numbers are conjectured only.

Enumeration results:

- Groups of order p^3:

p	$g(p^3)$	$f(p^3)$
2	2	1
≥ 3	2	0

- Groups of order p^4:

p	$g(p^4)$	$f(p^4)$
2	9	2
≥ 3	10	1

- Groups of order p^5:

p	$g(p^5)$	$f(p^5)$
2	44	5
3	60	5
≥ 5	$2p + 2\gcd(p-1,3) + \gcd(p-1,4) + 54$	$4 + \gcd(p-1,4)$

- Groups of order p^6:

p	$g(p^6)$	$f(p^6)$
2	256	13
3	493	12
5	673	25
≥ 7	$3p^2 + 39p + 24\gcd(p-1,3) + 11\gcd(p-1,4) + 2\gcd(p-1,5) + 333$	$11 + 2(\gcd(p-1,3) + \gcd(p-1,4) + \gcd(p-1,5))$

For fixed primes p the numbers $f(p^n)$ can be computed readily using the methods of Chapter 5.

6.7. Remark: We remark at the conjectural enumeration suggests that the numbers of p-groups G with $w(G) = Z_2(G)$ is small compared with the overall number of all groups of the considered order. For all prime powers p^n with $n \leq 6$ the number $f(p^n)$ seems to be linear in p.

Appendix A

Presentations of groups of order dividing p^6 with large Wielandt subgroup

This chapter is based on the enumeration of Section 6.2. Here we exhibit the presentations of the finite non-abelian p-groups (up to isomorphism) of order p^n satisfying that $w(G) = Z_2(G)$ for $n \leq 6$. We calculated the center, second center of all these groups and the index of the center in the second center for $p \geq 5$. Further all these presentations are polycyclic presentations. We omit the trivial power and commutator relations, for details we refer to [21, page 278 ff].

Groups of order p^3.
(i) If $p = 2$, then we have the following well known group.

Group	Center	Second center
$G = Q_8$	C_2	G

For $p > 2$ there does not exist any group G with $w(G) = Z_2(G)$.

Groups of order p^4.
(i) If $p = 2$, then we have the following well known groups.

Group	Center	Second center
$G = Q_{16}$	C_2	C_4
$G = Q_8 \times C_2$	$C_2 \times C_2$	G

(ii) If $p = 3$, then there exists only one group generated by $g_1, \ldots, g_4$.

Commutator relations	Power relations	Center	Second center
$[g_2, g_1] = g_3, [g_3, g_1] = g_4$	$g_1^p = g_4, g_2^p = g_4^2$	$\langle g_4 \rangle$	$\langle g_3, g_4 \rangle$

(iii) If $p \geq 5$, then there exists only one group generated by $g_1, \ldots, g_4$.

Commutator relations	Power relations	Center	Second center	$[Z_2(G) : Z(G)]$
$[g_2, g_1] = g_3, [g_3, g_1] = g_4$	$g_1^p = g_4$	$\langle g_4 \rangle$	$\langle g_3, g_4 \rangle$	p

Groups of order p^5.
(i) If $p = 2$, then we have the following groups generated by $g_1, \ldots, g_5$.

Commutator relations	Power relations	Center	Second center
$[g_2, g_1] = g_3, [g_3, g_1] = g_5,$ $[g_4, g_2] = g_5$	$g_1^p = g_4, g_2^p = g_5,$ $g_4^p = g_5$	$\langle g_5 \rangle$	$\langle g_3, g_4, g_5 \rangle$
$[g_2, g_1] = g_3, [g_3, g_1] = g_5$	$g_1^p = g_4, g_2^p = g_3,$ $g_3^p = g_5, g_4^p = g_5$	$\langle g_4, g_5 \rangle$	$\langle g_3, g_4, g_5 \rangle$
$[g_2, g_1] = g_3, [g_3, g_1] = g_4,$ $[g_3, g_2] = g_4, [g_4, g_1] = g_5,$ $[g_4, g_2] = g_5$	$g_1^p = g_5, g_2^p = g_5,$ $g_3^p = g_4 g_5,\ g_4^p = g_5$	$\langle g_5 \rangle$	$\langle g_4, g_5 \rangle$
$[g_2, g_1] = g_4, [g_4, g_1] = g_5,$ $[g_4, g_2] = g_5$	$g_1^p = g_5, g_2^p = g_5,$ $g_4^p = g_5$	$\langle g_3, g_5 \rangle$	$\langle g_3, g_4, g_5 \rangle$
$[g_2, g_1] = g_5$	$g_1^p = g_5, g_2^p = g_5$	$\langle g_3, g_4, g_5 \rangle$	G

(ii) If $p = 3$, then we have the following groups generated by $g_1, \ldots, g_5$.

Commutator relations	Power relations	Center	Second center
$[g_2, g_1] = g_3, [g_3, g_1] = g_5$	$g_1^p = g_4, g_4^p = g_5$	$\langle g_4, g_5 \rangle$	$\langle g_3, g_4, g_5 \rangle$
$[g_2, g_1] = g_3, [g_3, g_1] = g_4,$ $[g_4, g_1] = g_5$	$g_1^p = g_5, g_2^p = g_4^2 g_5, g_3^p = g_5^2$	$\langle g_5 \rangle$	$\langle g_4, g_5 \rangle$
$[g_2, g_1] = g_3, [g_3, g_1] = g_4,$ $[g_3, g_2] = g_5, [g_4, g_1] = g_5$	$g_1^p = g_5, g_2^p = g_4^2, g_3^p = g_5^2$	$\langle g_5 \rangle$	$\langle g_4, g_5 \rangle$
$[g_2, g_1] = g_4, [g_4, g_1] = g_5$	$g_1^p = g_5, g_2^p = g_5^2$	$\langle g_3, g_5 \rangle$	$\langle g_3, g_4, g_5 \rangle$
$[g_2, g_1] = g_4, [g_3, g_2] = g_5,$ $[g_4, g_1] = g_5$	$g_1^p = g_5, g_2^p = g_5^2$	$\langle g_5 \rangle$	$\langle g_3, g_4, g_5 \rangle$

(iii) If $p \geq 5$, then we have the following groups generated by $g_1, \dots, g_5$. Further let w be a generator of the multiplicative group of $\mathbb{F}_p$ and $b \in W_4 = \{x \in \mathbb{F}_p \mid x^4 = 1\}$.

Commutator relations	Power relations	Center	Second center	$[Z_2(G) : Z(G)]$
$[g_2, g_1] = g_3, [g_3, g_1] = g_4,$ $[g_3, g_2] = g_5$	$g_1^p = g_4, g_2^p = g_5$	$\langle g_4, g_5 \rangle$	$\langle g_3, g_4, g_5 \rangle$	p
$[g_2, g_1] = g_3, [g_3, g_1] = g_5$	$g_1^p = g_4, g_4^p = g_5$	$\langle g_4, g_5 \rangle$	$\langle g_3, g_4, g_5 \rangle$	p
$[g_2, g_1] = g_3, [g_3, g_1] = g_4,$ $[g_4, g_1] = g_5$	$g_1^p = g_5$	$\langle g_5 \rangle$	$\langle g_4, g_5 \rangle$	p
$[g_2, g_1] = g_3, [g_3, g_1] = g_4,$ $[g_4, g_1] = g_5^{w^b},$ $[g_3, g_2] = g_5^{w^b}$	$g_1^p = g_5$	$\langle g_5 \rangle$	$\langle g_4, g_5 \rangle$	p
$[g_2, g_1] = g_4, [g_4, g_2] = g_5$	$g_2^p = g_5$	$\langle g_3, g_5 \rangle$	$\langle g_3, g_4, g_5 \rangle$	p

Groups of order p^6.

(i) If $p = 2$, then we have the following groups generated by $g_1, \dots, g_6$.

Commutator relations	Power relations	Center	Second center
$[g_2, g_1] = g_3, [g_3, g_1] = g_6,$ $[g_5, g_1] = g_6$	$g_1^p = g_4, g_2^p = g_5,$ $g_3^p = g_6, g_4^p = g_6,$ $g_5^p = g_6$	$\langle g_4, g_3g_5, g_6 \rangle$	$\langle g_4, g_3g_5, g_5, g_6 \rangle$
$[g_2, g_1] = g_3, [g_3, g_1] = g_5,$ $[g_3, g_2] = g_6,$ $[g_4, g_2] = g_5g_6,$ $[g_4, g_3] = g_6, [g_5, g_1] = g_6$	$g_1^p = g_4, g_3^p = g_6,$ $g_4^p = g_6$	$\langle g_6 \rangle$	$\langle g_5, g_6 \rangle$
$[g_2, g_1] = g_3, [g_3, g_1] = g_5,$ $[g_3, g_2] = g_6,$ $[g_4, g_2] = g_5g_6,$ $[g_4, g_3] = g_6, [g_5, g_1] = g_6$	$g_1^p = g_4, g_2^p = g_6,$ $g_3^p = g_6, g_4^p = g_6$	$\langle g_6 \rangle$	$\langle g_5, g_6 \rangle$
$[g_2, g_1] = g_3, [g_3, g_1] = g_5,$ $[g_3, g_2] = g_5g_6,$ $[g_4, g_2] = g_6,$ $[g_5, g_1] = g_6, [g_5, g_2] = g_6$	$g_1^p = g_4, g_2^p = g_6,$ $g_3^p = g_5, g_4^p = g_6,$ $g_5^p = g_6$	$\langle g_6 \rangle$	$\langle g_4, g_5, g_6 \rangle$
$[g_2, g_1] = g_3, [g_3, g_1] = g_6$	$g_1^p = g_4, g_2^p = g_3,$ $g_3^p = g_6, g_4^p = g_5,$ $g_5^p = g_6$	$\langle g_4, g_5, g_6 \rangle$	$\langle g_3, g_4, g_5, g_6 \rangle$
$[g_2, g_1] = g_3, [g_3, g_1] = g_5,$ $[g_5, g_1] = g_6$	$g_1^p = g_4,$ $g_2^p = g_3g_5,$ $g_4^p = g_6, g_5^p = g_6$	$\langle g_4, g_6 \rangle$	$\langle g_4, g_5, g_6 \rangle$

Commutator relations	Power relations	Center	Second center
$[g_2,g_1]=g_3,[g_3,g_1]=g_4,$ $[g_3,g_2]=g_4,[g_4,g_1]=g_5,$ $[g_4,g_2]=g_5,[g_5,g_1]=g_6,$ $[g_5,g_2]=g_6$	$g_1^p=g_6,g_2^p=g_6,$ $g_3^p=g_4g_5,$ $g_4^p=g_5g_6,$ $g_5^p=g_6$	$\langle g_6\rangle$	$\langle g_5,g_6\rangle$
$[g_2,g_1]=g_4,[g_4,g_1]=g_6,$ $[g_5,g_2]=g_6$	$g_1^p=g_5,g_2^p=g_6,$ $g_5^p=g_6$	$\langle g_3,g_6\rangle$	$\langle g_3,g_4,g_5,g_6\rangle$
$[g_2,g_1]=g_4,[g_4,g_1]=g_6$	$g_1^p=g_5,g_2^p=g_4,$ $g_4^p=g_6,g_5^p=g_6$	$\langle g_3,g_5,g_6\rangle$	$\langle g_3,g_4,g_5,g_6\rangle$
$[g_2,g_1]=g_4,[g_4,g_1]=g_6,$ $[g_3,g_2]=g_6$	$g_1^p=g_5,g_2^p=g_4,$ $g_4^p=g_6,g_5^p=g_6$	$\langle g_5,g_6\rangle$	$\langle g_3,g_4,g_5,g_6\rangle$
$[g_2,g_1]=g_4,[g_4,g_1]=g_5,$ $[g_4,g_2]=g_5,[g_5,g_1]=g_6,$ $[g_5,g_2]=g_6$	$g_1^p=g_6,g_2^p=g_6,$ $g_4^p=g_5g_6,$ $g_5^p=g_6$	$\langle g_3,g_6\rangle$	$\langle g_3,g_5,g_6\rangle$
$[g_2,g_1]=g_5,[g_5,g_1]=g_6,$ $[g_5,g_2]=g_6$	$g_1^p=g_6,g_2^p=g_6,$ $g_5^p=g_6$	$\langle g_3,g_4,g_6\rangle$	$\langle g_3,g_4,g_5,g_6\rangle$
$[g_2,g_1]=g_6$	$g_1^p=g_6,g_2^p=g_6$	$\langle g_3,g_4,g_5,g_6\rangle$	$\langle g_1,g_2,g_3,g_4,g_5,g_6\rangle$

(ii) If $p=3$, then we have the following groups generated by $g_1,\ldots,g_6$.

Commutator relations	Power relations	Center	Second center
$[g_2,g_1]=g_3,[g_3,g_1]=g_6$	$g_1^p=g_4,g_2^p=g_5,$ $g_3^p=g_6$	$\langle g_4,g_5,g_6\rangle$	$\langle g_3,g_4,g_5,g_6\rangle$
$[g_2,g_1]=g_3,[g_3,g_1]=g_6$	$g_1^p=g_4,g_4^p=g_5,$ $g_5^p=g_6$	$\langle g_4,g_5,g_6\rangle$	$\langle g_3,g_4,g_5,g_6\rangle$
$[g_2,g_1]=g_3,[g_3,g_1]=g_5,$ $[g_5,g_1]=g_6$	$g_1^p=g_4,g_2^p=g_5,$ $g_3^p=g_6,g_4^p=g_6$	$\langle g_4,g_6\rangle$	$\langle g_4,g_5,g_6\rangle$
$[g_2,g_1]=g_3,[g_3,g_1]=g_5,$ $[g_3,g_2]=g_6,[g_5,g_1]=g_6$	$g_1^p=g_4,g_2^p=g_5^2,$ $g_3^p=g_6^2,g_4^p=g_6$	$\langle g_4,g_6\rangle$	$\langle g_4,g_5,g_6\rangle$
$[g_2,g_1]=g_3,[g_3,g_1]=g_5,$ $[g_3,g_2]=g_6,[g_5,g_1]=g_6$	$g_1^p=g_4,g_2^p=g_5^2,$ $g_3^p=g_6^2,g_4^p=g_6^2$	$\langle g_4,g_6\rangle$	$\langle g_4,g_5,g_6\rangle$
$[g_2,g_1]=g_3,[g_3,g_1]=g_4,$ $[g_4,g_1]=g_5,[g_5,g_1]=g_6$	$g_1^p=g_6,g_2^p=g_4^2g_5,$ $g_3^p=g_5^2g_6,g_4^p=g_6^2$	$\langle g_6\rangle$	$\langle g_5,g_6\rangle$
$[g_2,g_1]=g_3,[g_3,g_1]=g_4,$ $[g_3,g_2]=g_6,[g_4,g_1]=g_5,$ $[g_5,g_1]=g_6$	$g_1^p=g_6^2,g_2^p=g_4^2g_5,$ $g_3^p=g_5^2g_6,g_4^p=g_6^2$	$\langle g_6\rangle$	$\langle g_5,g_6\rangle$
$[g_2,g_1]=g_4,[g_4,g_1]=g_6$	$g_1^p=g_5,g_5^p=g_6$	$\langle g_3,g_5,g_6\rangle$	$\langle g_3,g_4,g_5,g_6\rangle$
$[g_2,g_1]=g_4,[g_4,g_1]=g_5,$ $[g_5,g_1]=g_6$	$g_1^p=g_6,g_2^p=g_5^2g_6,$ $g_4^p=g_6^2$	$\langle g_3,g_6\rangle$	$\langle g_3,g_5,g_6\rangle$
$[g_2,g_1]=g_4,[g_4,g_1]=g_5,$ $[g_4,g_2]=g_6$ $[g_5,g_1]=g_6$	$g_1^p=g_6,g_2^p=g_5^2,$ $g_4^p=g_6^2$	$\langle g_3,g_6\rangle$	$\langle g_3,g_5,g_6\rangle$

Commutator relations	Power relations	Center	Second center
$[g_2, g_1] = g_5, [g_5, g_1] = g_6$	$g_1^p = g_6, g_2^p = g_6^2$	$\langle g_3, g_4, g_6\rangle$	$\langle g_3, g_4, g_5, g_6\rangle$
$[g_2, g_1] = g_5, [g_3, g_2] = g_6,$ $[g_5, g_1] = g_6$	$g_1^p = g_6, g_2^p = g_6^2$	$\langle g_4, g_6\rangle$	$\langle g_3, g_4, g_5, g_6\rangle$

(iii) If $p = 5$, then we have the following groups generated by $g_1, \ldots, g_6$.

Commutator relations	Power relations	Center	Second center	$[Z_2(G) : Z(G)]$
$[g_2, g_1] = g_3, [g_3, g_2] = g_6^4$	$g_2^p = g_4, g_4^p = g_5,$ $g_5^p = g_6$	$\langle g_4, g_5, g_6\rangle$	$\langle g_3, g_4, g_5, g_6\rangle$	p
$[g_2, g_1] = g_4, [g_2, g_4] = g_6$	$g_2^p = g_5, g_5^p = g_6^4$	$\langle g_3, g_5, g_6\rangle$	$\langle g_3, g_4, g_5, g_6\rangle$	p
$[g_2, g_1] = g_3, [g_3, g_2] = g_6$	$g_1^p = g_4, g_2^p = g_5,$ $g_5^p = g_6^4$	$\langle g_4, g_5, g_6\rangle$	$\langle g_3, g_4, g_5, g_6\rangle$	p
$[g_2, g_1] = g_5, [g_5, g_2] = g_6$	$g_2^p = g_6^4$	$\langle g_3, g_4, g_6\rangle$	$\langle g_3, g_4, g_5, g_6\rangle$	p
$[g_2, g_1] = g_4, [g_4, g_1] = g_6,$ $[g_4, g_2] = g_5, [g_5, g_2] = g_6$	$g_2^p = g_6^2$	$\langle g_3, g_6\rangle$	$\langle g_3, g_5, g_6\rangle$	p
$[g_2, g_1] = g_3, [g_3, g_1] = g_4,$ $[g_3, g_2] = g_5, [g_5, g_2] = g_6$	$g_2^p = g_6^4$	$\langle g_4, g_6\rangle$	$\langle g_4, g_5, g_6\rangle$	p
$[g_2, g_1] = g_5, [g_4, g_1] = g_6^4,$ $[g_3, g_2] = g_4, [g_4, g_3] = g_5,$ $[g_5, g_3] = g_6$	$g_3^p = g_6^4$	$\langle g_6\rangle$	$\langle g_5, g_6\rangle$	p
$[g_2, g_1] = g_3, [g_3, g_2] = g_4,$ $[g_4, g_2] = g_5, [g_5, g_2] = g_6$	$g_1^p = g_6, g_2^p = g_6^4$	$\langle g_6\rangle$	$\langle g_5, g_6\rangle$	p
$[g_2, g_1] = g_3, [g_3, g_1] = g_6,$ $[g_3, g_2] = g_4, [g_4, g_2] = g_5,$ $[g_5, g_2] = g_6$	$g_1^p = g_6, g_2^p = g_6^4$	$\langle g_6\rangle$	$\langle g_5, g_6\rangle$	p
$[g_2, g_1] = g_4, [g_4, g_1] = g_5,$ $[g_4, g_2] = g_6$	$g_1^p = g_5^4, g_2^p = g_6^4$	$\langle g_3, g_5, g_6\rangle$	$\langle g_3, g_4, g_5, g_6\rangle$	p
$[g_2, g_1] = g_3, [g_3, g_2] = g_5,$ $[g_5, g_2] = g_6$	$g_2^p = g_4, g_4^p = g_6^4$	$\langle g_4, g_6\rangle$	$\langle g_4, g_5, g_6\rangle$	p
$[g_2, g_1] = g_4, [g_4, g_2] = g_5,$ $[g_5, g_2] = g_6$	$g_2^p = g_6^4$	$\langle g_3, g_6\rangle$	$\langle g_3, g_5, g_6\rangle$	p
$[g_2, g_1] = g_3, [g_3, g_1] = g_6,$ $[g_3, g_2] = g_5, [g_5, g_2] = g_6$	$g_2^p = g_4, g_4^p = g_6^4$	$\langle g_4, g_6\rangle$	$\langle g_4, g_5, g_6\rangle$	p
$[g_2, g_1] = g_3, [g_3, g_1] = g_6,$ $[g_3, g_2] = g_5, [g_5, g_2] = g_6$	$g_2^p = g_4, g_4^p = g_6^3$	$\langle g_4, g_6\rangle$	$\langle g_4, g_5, g_6\rangle$	p
$[g_2, g_1] = g_3, [g_3, g_1] = g_6,$ $[g_3, g_2] = g_5, [g_5, g_2] = g_6$	$g_2^p = g_4, g_4^p = g_6$	$\langle g_4, g_6\rangle$	$\langle g_4, g_5, g_5\rangle$	p
$[g_2, g_1] = g_3, [g_3, g_1] = g_6,$ $[g_3, g_2] = g_5, [g_5, g_2] = g_6$	$g_2^p = g_4, g_4^p = g_6^2$	$\langle g_4, g_6\rangle$	$\langle g_4, g_5, g_6\rangle$	p

Commutator relations	Power relations	Center	Second center	$[Z_2(G):Z(G)]$
$[g_2,g_1]=g_4, [g_4,g_1]=g_6, [g_4,g_2]=g_5, [g_5,g_2]=g_6$	$g_2^p=g_6^4$	$\langle g_3,g_6\rangle$	$\langle g_3,g_5,g_6\rangle$	p
$[g_2,g_1]=g_4, [g_4,g_1]=g_6, [g_4,g_2]=g_5, [g_5,g_2]=g_6$	$g_2^p=g_6^3$	$\langle g_3,g_6\rangle$	$\langle g_3,g_5,g_6\rangle$	p
$[g_2,g_1]=g_4, [g_4,g_1]=g_6, [g_4,g_2]=g_5, [g_5,g_2]=g_6$	$g_2^p=g_6$	$\langle g_3,g_6\rangle$	$\langle g_3,g_5,g_6\rangle$	p
$[g_2,g_1]=g_3, [g_3,g_1]=g_6, [g_3,g_2]=g_4, [g_4,g_2]=g_5, [g_5,g_2]=g_6$	$g_1^p=g_6, g_2^p=g_6^3$	$\langle g_6\rangle$	$\langle g_5,g_6\rangle$	p
$[g_2,g_1]=g_3, [g_4,g_1]=g_6, [g_5,g_1]=g_6, [g_3,g_2]=g_4, [g_4,g_2]=g_5, [g_4,g_3]=g_6$	$g_1^p=g_6^4, g_2^p=g_6^3$	$\langle g_6\rangle$	$\langle g_5,g_6\rangle$	p
$[g_2,g_1]=g_3, [g_4,g_1]=g_6, [g_5,g_1]=g_6, [g_3,g_2]=g_4, [g_4,g_2]=g_5, [g_4,g_3]=g_6$	$g_1^p=g_6^4, g_2^p=g_6$	$\langle g_6\rangle$	$\langle g_5,g_6\rangle$	p
$[g_2,g_1]=g_3, [g_3,g_1]=g_5, [g_4,g_1]=g_6, [g_3,g_2]=g_4, [g_4,g_2]=g_5, [g_5,g_2]=g_6$	$g_1^p=g_6, g_2^p=g_6^4$	$\langle g_6\rangle$	$\langle g_5,g_6\rangle$	p
$[g_2,g_1]=g_3, [g_3,g_1]=g_5, [g_4,g_1]=g_6, [g_5,g_1]=g_6, [g_3,g_2]=g_4, [g_4,g_2]=g_5, [g_4,g_3]=g_6$	$g_1^p=g_6^4, g_2^p=g_6^3$	$\langle g_6\rangle$	$\langle g_5,g_6\rangle$	p
$[g_2,g_1]=g_3, [g_3,g_1]=g_5, [g_4,g_1]=g_6, [g_5,g_1]=g_6, [g_3,g_2]=g_4, [g_4,g_2]=g_5, [g_4,g_3]=g_6$	$g_1^p=g_6^4, g_2^p=g_6$	$\langle g_6\rangle$	$\langle g_5,g_6\rangle$	p

(iv) If $p\geq 7$, then we have the following groups generated by $g_1,\ldots,g_6$. Where $k\in\{0,1,2,3,..,p-1\}$ and w is a generator of the multiplicative group of $\mathbb{F}_p$.

Commutator relations	Power relations	Center	Second center	$[Z_2(G):Z(G)]$
$[g_2,g_1]=g_3g_6^{(p+1)/2}$, $[g_3,g_1]=g_6$	$g_1^p=g_4$, $g_2^p=g_5$, $g_4^p=g_6$	$\langle g_4,g_5,g_6\rangle$	$\langle g_3,g_4,g_5,g_6\rangle$	p
$[g_2,g_1]=g_3g_4^{(p+1)/2}g_5^{(p+1)/2}g_6^k$, $[g_3,g_1]=g_3g_4g_6^{(p+1)/2}$, $[g_4,g_1]=g_6$, $[g_3,g_2]=g_5$	$g_1^p=g_6$	$\langle g_5,g_6\rangle$	$\langle g_4,g_5,g_6\rangle$	p

Commutator relations	Power relations	Center	Second center	$[Z_2(G):Z(G)]$
$[g_2,g_1]=g_3g_6^{(p+1)/2}$, $[g_3,g_1]=g_6$	$g_1^p=g_4$, $g_4^p=g_5$, $g_5^p=g_6$	$\langle g_4,g_5,g_6\rangle$	$\langle g_3,g_4,g_5,g_6\rangle$	p
$[g_2,g_1]=g_3g_5^{(p+1)/2}g_6^k$, $[g_3,g_1]=g_5g_6^{(p+1)/2}$, $[g_5,g_1]=g6$	$g_1^p=g_4$, $g_4^p=g_6$	$\langle g_4,g_6\rangle$	$\langle g_4,g_5,g_6\rangle$	p
$[g_2,g_1]=g_3g_5^{(p+1)/2}g_6^k$, $[g_3,g_1]=g_5g_6^{(p+1)/2}$, $[g_5,g_1]=g_6$, $[g_3,g_2]=g_6$	$g_1^p=g_4$, $g_4^p=g_6$	$\langle g_4,g_6\rangle$	$\langle g_4,g_5,g_6\rangle$	p
$[g_2,g_1]=g_3g_5^{(p+1)/2}g_6^k$, $[g_3,g_1]=g_5g_6^{(p+1)/2}$, $[g_5,g_1]=g_6$, $[g_3,g_2]=g_6^w$	$g_1^p=g_4$, $g_4^p=g_6$	$\langle g_4,g_6\rangle$	$\langle g_4,g_5,g_6\rangle$	p
$[g_2,g_1]=g_3g_4^{(p+1)/2}g_5^kg_6^k$, $[g_3,g_1]=g_4g_5^{(p+1)/2}g_6^k$, $[g_4,g_1]=g_5g_6^{(p+1)/2}$, $[g_5,g_1]=g_6$	$g_1^p=g_6$	$\langle g_6\rangle$	$\langle g_5,g_6\rangle$	p
$[g_2,g_1]=g_3g_4^{(p+1)/2}g_5^kg_6^k$, $[g_3,g_1]=g_4g_5^{(p+1)/2}g_6^k$, $[g_4,g_1]=g_5g_6^{(p+1)/2}$, $[g_5,g_1]=g_6,[g_3,g_2]=g_6$	$g_1^p=g_6$	$\langle g_6\rangle$	$\langle g_5,g_6\rangle$	p
$[g_2,g_1]=g_3g_4^{(p+1)/2}g_5^kg_6^k$, $[g_3,g_1]=g_4g_5^{(p+1)/2}g_6^k$, $[g_4,g_1]=g_5g_6^{(p+1)/2},[g_5,g_1]=g_6$	$g_1^p=g_6^w$	$\langle g_6\rangle$	$\langle g_5,g_6\rangle$	p
$[g_2,g_1]=g_3g_4^{(p+1)/2}g_5^kg_6^k$, $[g_3,g_1]=g_4g_5^{(p+1)/2}g_6^k$, $[g_4,g_1]=g_5,[g_5,g_2]=g_6$, $[g_4,g_3]=g_6^{(p+1)/2}$	$g_2^p=g_6$	$\langle g_6\rangle$	$\langle g_5,g_6\rangle$	p
$[g_2,g_1]=g_3g_4^{(p+1)/2}g_5^k$, $[g_3,g_1]=g_4g_5^{(p+1)/2}g_6^k$, $[g_4,g_1]=g_5$, $[g_3,g_2]=g_5g_6^{(p+1)/2},[g_5,g_2]=g_6$, $[g_4,g_3]=g_6^{(p-1)}$	$g_2^p=g_6$	$\langle g_6\rangle$	$\langle g_5,g_6\rangle$	p

Commutator relations	Power relations	Center	Second center	$[Z_2(G):Z(G)]$
$[g_2,g_1]=g_3g_4^{(p+1)/2}g_5^kg_6^k$, $[g_3,g_1]=g_4g_5^{(p+1)/2}g_6^k$, $[g_4,g_1]=g_5g_6^{(p+1)/2}$, $[g_5,g_1]=g_6, [g_3,g_2]=g_5$, $[g_4,g_2]=g_6$	$g_1^p=g_6$	$\langle g_6\rangle$	$\langle g_5,g_6\rangle$	p
$[g_2,g_1]=g_4g_5^{(p+1)/2}g_6^{(p+1)/2}$, $[g_4,g_1]=g_5, [g_4,g_2]=g_6$	$g_1^p=g_5$, $g_2^p=g_6$	$\langle g_3,g_5,g_6\rangle$	$\langle g_3,g_4,g_5,g_6\rangle$	p
$[g_2,g_1]=g_4g_6^{(p+1)/2}$, $[g_4,g_1]=g_6$	$g_1^p=g_5$, $g_5^p=g_6$	$\langle g_3,g_5,g_6\rangle$	$\langle g_3,g_4,g_5,g_6\rangle$	p
$[g_2,g_1]=g_4g_5^{(p+1)/2}g_6^k$, $[g_4,g_1]=g_5g_6^{(p+1)/2}, [g_5,g_1]=g_6$	$g_1^p=g_6$	$\langle g_3,g_6\rangle$	$\langle g_3,g_5,g_6\rangle$	p
$[g_2,g_1]=g_4g_5^{(p+1)/2}g_6^k$, $[g_4,g_1]=g_5g_6^{(p+1)/2}, [g_5,g_1]=g_6$, $[g_4,g_2]=g_6$	$g_1^p=g_6$	$\langle g_3,g_6\rangle$	$\langle g_3,g_5,g_6\rangle$	p
$[g_2,g_1]=g_4g_5^{(p+1)/2}g_6^k$, $[g_4,g_1]=g_5g_6^{(p+1)/2}$, $[g_5,g_1]=g_6, [g_4,g_2]=g_6^w$	$g_1^p=g_6$	$\langle g_3,g_6\rangle$	$\langle g_3,g_5,g_6\rangle$	p
$[g_2,g_1]=g_4g_5^{(p+1)/2}g_6^k$, $[g_4,g_1]=g_5g_6^{(p+1)/2}, [g_5,g_1]=g_6$, $[g_3,g_2]=g_5, [g_4,g_3]=g_6^{(p-1)}$	$g_1^p=g_6$	$\langle g_6\rangle$	$\langle g_5,g_6\rangle$	p
$[g_2,g_1]=g_5g_6^{(p+1)/2}$, $[g_5,g_1]=g_6$	$g_1^p=g_6$	$\langle g_3,g_4,g_6\rangle$	$\langle g_3,g_4,g_5,g_6\rangle$	p

(v) If $p\equiv 1 \bmod 3$ only, then we have the following groups generated by $g_1,\dots,g_6$ and w is a generator of the multiplicative group of $\mathbb{F}_p$.

Commutator relations	Power relations	Center	Second center	$[Z_2(G):Z(G)]$
$[g_2,g_1]=g_3g_4^{(p+1)/2}g_5^{(p-1)}g_6^{(w-1)}$, $[g_3,g_1]=g_4g_5^{(p+1)/2}g_6^{(p-1)}$, $[g_4,g_1]=g_5g_6^{(p+1)/2}, [g_5,g_1]=g_6$, $[g_3,g_2]=g_6$	$g_1^p=g_6^2$	$\langle g_6\rangle$	$\langle g_5,g_6\rangle$	p

Commutator relations	Power relations	Center	Second center	$[Z_2(G):Z(G)]$
$[g_2,g_1]=g_3g_4^{(p+1)/2}g_5^{(p-1)}g_6^{(w-1)}$, $[g_3,g_1]=g_4g_5^{(p+1)/2}g_6^{(p-1)}$, $[g_4,g_1]=g_5g_6^{(p+1)/2}$, $[g_5,g_1]=g_6$, $[g_3,g_2]=g_6$	$g_1^p=g_6^6$	$\langle g_6\rangle$	$\langle g_5,g_6\rangle$	p
$[g_2,g_1]=g_3g_4^{(p+1)/2}g_5^{(p-1)}g_6^{(w-1)}$, $[g_3,g_1]=g_4g_5^{(p+1)/2}g_6^{(p-1)}$, $[g_4,g_1]=g_5g_6^{(p+1)/2}$, $[g_5,g_1]=g_6$, $[g_3,g_2]=g_6$	$g_1^p=g_6^4$	$\langle g_6\rangle$	$\langle g_5,g_6\rangle$	p
$[g_2,g_1]=g_3g_4^{(p+1)/2}g_5^{(p-1)}g_6^{(w-1)}$, $[g_3,g_1]=g_4g_5^{(p+1)/2}g_6^{(p-1)}$, $[g_4,g_1]=g_5g_6^{(p+1)/2}$, $[g_5,g_1]=g_6$, $[g_3,g_2]=g_6$	$g_1^p=g_6^5$	$\langle g_6\rangle$	$\langle g_5,g_6\rangle$	p

(vi) If $p\equiv 1 \bmod 5$ only, then we have the following groups generated by $g_1,\dots,g_6$ and w is a generator of the multiplicative group of $\mathbb{F}_p$.

Commutator relations	Power relations	Center	Second center	$[Z_2(G):Z(G)]$
$[g_2,g_1]=g_3g_4^{(p+1)/2}g_5^8$, $[g_3,g_1]=g_4g_5^{(p+1)/2}g_6^{(p-1)/2}$, $[g_4,g_1]=g_5$, $[g_3,g_2]=g_5g_6^{(p+1)/2}$, $[g_5,g_2]=g_6$, $[g_4,g_3]=g_6^{(p-1)}$	$g_2^p=g_6^2$	$\langle g_6\rangle$	$\langle g_5,g_6\rangle$	p
$[g_2,g_1]=g_3g_4^{(p+1)/2}g_5^8$, $[g_3,g_1]=g_4g_5^{(p+1)/2}g_6^{(p-1)/2}$, $[g_4,g_1]=g_5$, $[g_3,g_2]=g_5g_6^{(p+1)/2}$ $[g_5,g_2]=g_6$, $[g_4,g_3]=g_6^{(p-1)}$	$g_2^p=g_6^4$	$\langle g_6\rangle$	$\langle g_5,g_6\rangle$	p
$[g_2,g_1]=g_3g_4^{(p+1)/2}g_5^8$, $[g_3,g_1]=g_4g_5^{(p+1)/2}g_6^{(p-1)/2}$, $[g_4,g_1]=g_5$, $[g_3,g_2]=g_5g_6^{(p+1)/2}$ $[g_5,g_2]=g_6$, $[g_4,g_3]=g_6^{(p-1)}$	$g_2^p=g_6^8$	$\langle g_6\rangle$	$\langle g_5,g_6\rangle$	p
$[g_2,g_1]=g_3g_4^{(p+1)/2}g_5^8$, $[g_3,g_1]=g_4g_5^{(p+1)/2}g_6^{(p-1)/2}$, $[g_4,g_1]=g_5$, $[g_3,g_2]=g_5g_6^{(p+1)/2}$, $[g_5,g_2]=g_6$, $[g_4,g_3]=g_6^{(p-1)}$	$g_2^p=g_6^5$	$\langle g_6\rangle$	$\langle g_5,g_6\rangle$	p

Commutator relations	Power relations	Center	Second center	$[Z_2(G):Z(G)]$
$[g_2,g_1]=g_3g_4^{(p+1)/2}g_5^8g_6^9$, $[g_3,g_1]=g_4g_5^{(p+1)/2}g_6^w$, $[g_4,g_1]=g_5g_6^{(p+1)/2}$, $[g_5,g_1]=g_6$ $[g_3,g_2]=g_5$, $[g_4,g_2]=g_6$	$g_1^p=g_6^2$	$\langle g_6\rangle$	$\langle g_5,g_6\rangle$	p
$[g_2,g_1]=g_3g_4^{(p+1)/2}g_5^8g_6^9$, $[g_3,g_1]=g_4g_5^{(p+1)/2}g_6^w$, $[g_4,g_1]=g_5g_6^{(p+1)/2}$, $[g_5,g_1]=g_6$ $[g_3,g_2]=g_5$, $[g_4,g_2]=g_6$	$g_1^p=g_6^4$	$\langle g_6\rangle$	$\langle g_5,g_6\rangle$	p
$[g_2,g_1]=g_3g_4^{(p+1)/2}g_5^8g_6^9$, $[g_3,g_1]=g_4g_5^{(p+1)/2}g_6^w$, $[g_4,g_1]=g_5g_6^{(p+1)/2}$, $[g_5,g_1]=g_6$ $[g_3,g_2]=g_5$, $[g_4,g_2]=g_6$	$g_1^p=g_6^8$	$\langle g_6\rangle$	$\langle g_5,g_6\rangle$	p
$[g_2,g_1]=g_3g_4^{(p+1)/2}g_5^8g_6^9$, $[g_3,g_1]=g_4g_5^{(p+1)/2}g_6^w$, $[g_4,g_1]=g_5g_6^{(p+1)/2}$, $[g_5,g_1]=g_6$, $[g_3,g_2]=g_5$, $[g_4,g_2]=g_6$	$g_1^p=g_6^5$	$\langle g_6\rangle$	$\langle g_5,g_6\rangle$	p

(vii) If $p\equiv 1 \bmod 3$ and $p\equiv 1 \bmod 4$ only, then we have the following groups generated by $g_1,\ldots,g_6$.

Commutator relations	Power relations	Center	Second center	$[Z_2(G):Z(G)]$
$[g_2,g_1]=g_3g_4^{(p+1)/2}g_5^{(p-2)}$, $[g_3,g_1]=g_4g_5^{(p+1)/2}g_6^{(p-2)}$, $[g_4,g_1]=g_5g_6^{(p+1)/2}$, $[g_5,g_1]=g_6$, $[g_3,g_2]=g_6$	$g_1^p=g_6^4$	$\langle g_6\rangle$	$\langle g_5,g_6\rangle$	p
$[g_2,g_1]=g_3g_4^{(p+1)/2}g_5^{(p-2)}$, $[g_3,g_1]=g_4g_5^{(p+1)/2}g_6^{(p-2)}$, $[g_4,g_1]=g_5g_6^{(p+1)/2}$, $[g_5,g_1]=g_6$, $[g_3,g_2]=g_6$	$g_1^p=g_6^8$	$\langle g_6\rangle$	$\langle g_5,g_6\rangle$	p

Commutator relations	Power relations	Center	Second center	$[Z_2(G):Z(G)]$
$[g_2,g_1]=g_3g_4^{(p+1)/2}g_5^{(p-2)}$, $[g_3,g_1]=g_4g_5^{(p+1)/2}g_6^{(p-2)}$, $[g_4,g_1]=g_5g_6^{(p+1)/2}$, $[g_5,g_1]=g_6,[g_3,g_2]=g_6$	$g_1^p=g_6^3$	$\langle g_6\rangle$	$\langle g_5,g_6\rangle$	p
$[g_2,g_1]=g_3g_4^{(p+1)/2}g_5^{(p-2)}$, $[g_3,g_1]=g_4g_5^{(p+1)/2}g_6^{(p-2)}$, $[g_4,g_1]=g_5g_6^{(p+1)/2}$, $[g_5,g_1]=g_6,[g_3,g_2]=g_6$	$g_1^p=g_6^6$	$\langle g_6\rangle$	$\langle g_5,g_6\rangle$	p
$[g_2,g_1]=g_3g_5^{(p+1)/2}$, $[g_3,g_1]=g_5g_6^{(p+1)/2}$, $[g_5,g_1]=g_6,[g_3,g_2]=g_6^4$	$g_1^p=g_4$	$\langle g_4,g_6\rangle$	$\langle g_4,g_5,g_6\rangle$	p
$[g_2,g_1]=g_3g_5^{(p+1)/2}g_6^2$, $[g_3,g_1]=g_5g_6^{(p+1)/2}$, $[g_5,g_1]=g_6,[g_3,g_2]=g_6^8$	$g_1^p=g_4$	$\langle g_4,g_6\rangle$	$\langle g_4,g_5,g_6\rangle$	p
$[g_2,g_1]=g_4g_5^{(p+1)/2}$, $[g_4,g_1]=g_5g_6^{(p+1)/2}$, $[g_5,g_1]=g_6,[g_4,g_2]=g_6^4$	$g_1^p=g_6$	$\langle g_3,g_6\rangle$	$\langle g_3,g_5,g_6\rangle$	p
$[g_2,g_1]=g_4g_5^{(p+1)/2}g_6^2$, $[g_4,g_1]=g_5g_6^{(p+1)/2}$, $[g_5,g_1]=g_6,[g_4,g_2]=g_6^8$	$g_1^p=g_6$	$\langle g_3,g_6\rangle$	$\langle g_3,g_5,g_6\rangle$	p

(viii) If $p\equiv 1 \mod 4$ only, then we have the following groups generated by $g_1,\ldots,g_6$.

Commutator relations	Power relations	Center	Second center	$[Z_2(G):Z(G)]$
$[g_2,g_1]=g_3g_5^{(p+1)/2}g_6^{16}$, $[g_3,g_1]=g_5g_6^{(p+1)/2}$, $[g_5,g_1]=g_6,[g_3,g_2]=g_6^{(p+1)/2}$	$g_1^p=g_4$	$\langle g_4,g_6\rangle$	$\langle g_4,g_5,g_6\rangle$	p
$[g_2,g_1]=g_3g_5^{(p+1)/2}g_6^8$, $[g_3,g_1]=g_5g_6^{(p+1)/2}$, $[g_5,g_1]=g_6,[g_3,g_2]=g_6^{10}$	$g_1^p=g_4$	$\langle g_4,g_6\rangle$	$\langle g_4,g_5,g_6\rangle$	p

Commutator relations	Power relations	Center	Second center	$[Z_2(G):Z(G)]$
$[g_2,g_1]=g_4g_5^{(p+1)/2}g_6^{16}$, $[g_4,g_1]=g_5g_6^{(p+1)/2}$, $[g_5,g_1]=g_6,[g_4,g_2]=g_6^{(p+1)/2}$	$g_1^p=g_6$	$\langle g_3,g_6\rangle$	$\langle g_3,g_5,g_6\rangle$	p
$[g_2,g_1]=g_4g_5^{(p+1)/2}g_6^{8}$, $[g_4,g_1]=g_5g_6^{(p+1)/2}$, $[g_5,g_1]=g_6,[g_4,g_2]=g_6^{10}$	$g_1^p=g_6$	$\langle g_3,g_6\rangle$	$\langle g_3,g_5,g_6\rangle$	p

Chapter 7

Summary and Outlook

7.1 Summary and Outlook

The Wielandt subgroup $w(G)$ and its generalisations are group theoretic concepts which have been introduced as generalisations of the center of a group G. Historically, a first idea in this direction is due to Baer (1935), who introduced the Norm $N(G)$. The advantage of Wielandt's subgroup over the Baer Norm is that it is non-trivial in a much wider class of groups.

In my MPhil Thesis [2], see also [3], a generalisation of the Wielandt subgroup is introduced. Let G be a group and N a normal subgroup of G. The generalised Wielandt subgroup $w_N(G)$ of G with respect to N is the intersection of the normalisers of the subnormal subgroups of G contained in N. A first aim of this thesis is to investigate the structure of the generalised Wielandt subgroup. It recalls a variation of the results of [3] for completeness and also corrects a mistake in [3].

We introduce a new generalisation of the Wielandt subgroup which we call the "relative Wielandt subgroup". The relative Wielandt subgroup $w_c(G)$ of G is the intersection of the normalisers of the cyclic subnormal subgroups of G. It is noted that $w_c(G)$ satisfies some properties which do not hold in case of the ordinary Wielandt subgroup and an example is given which shows that $w_c(G)$ is non-trivial in a wider class of groups than $w(G)$.

Further, we develop algorithms to compute the Norm of a group, Wielandt subgroup and its variations and, based on this, we investigate the groups of order dividing p^6 with maximal Wielandt subgroup.

We describe the results of this thesis in more detail in the following.

The generalised Wielandt subgroup and its length

The following theorem completely characterises the higher order generalised Wielandt subgroup and its length.

Theorem: *Let G be a finite group and $N \trianglelefteq G$.*

(i) *Let $i \geqslant 0$, then $w_{N,i+1}(G) = \cap\{N_G(K) \mid w_{N,i}(G) \leq K \leq Nw_{N,i}(G), K \lhd\lhd G\}$.*

(ii) *Let $m, n \geq 0$ be integers. If the generalised Wielandt length of $G/w_{N,m}(G)$ with respect to $Nw_{N,m}(G)/w_{N,m}(G)$ is n, then $wl_N(G) = m + n$.*

The following theorem gives a structural description of soluble groups with generalised Wielandt length one.

Theorem: *Every soluble group G with $wl_{Fit(G)}(G) = 1$ is metabelian.*

The relative Wielandt subgroup

We introduce and investigate the relative Wielandt subgroup $w_c(G)$. Since $N(G) \leq w(G) \leq w_c(G)$ holds for every group G, $w_c(G)$ is always non-trivial for every finite group G.
In the following theorem we exhibit some properties of the relative Wielandt subgroup for polycyclic groups.

Theorem: *Let G be a polycyclic group.*

(i) *$w_c(G)Z(Fit(G))/Z(Fit(G)) \cong w_c(G)/w_c(G) \cap Z(Fit(G))$ is finite abelian.*

(ii) *$w_c(G)$ is supersoluble and metabelian.*

The equality in the following theorem does not hold for the Wielandt subgroup and its local version.

Theorem: *Let G and H be a groups. Then*

(i) *$w_c(G \times H) = w_c(G) \times w_c(H)$.*

(ii) *$w_c^p(G \times H) = w_c^p(G) \times w_c^p(H)$.*

Example: *Let D_∞ be the infinite dihedral group. Then $w(D_\infty) = \{1\}$ and $w_c(D_\infty) = D_\infty$. Hence the ordinary Wielandt subgroup and the relative Wielandt subgroup can differ.*

Algorithms

We developed algorithms to compute the Norm of a group G for a finite or a polycyclic group G. We also introduced methods to determine the ordinary, generalised or relative Wielandt subgroup of a finite group G. We note that there is no method available to compute the Wielandt subgroup of an arbitrary infinite polycyclic group at current; so this is an open problem. Further, for finite groups, we give a method to check that whether the given group is a T-group or not.

p-Groups with maximal Wielandt subgroup

A finite p-group G is nilpotent and hence by Schenkman [32] we have that $Z(G) \leq w(G) = N(G) \leq Z_2(G)$. Thus it seems interesting to try to understand the extremal case; that is, those p-groups G with $w(G) = Z_2(G)$.
We obtain a partial answer to the question raised by Guo [19]; about the structure of groups G with $w(G) = Z_2(G)$, as follows.

Theorem: *Let G be a finite p-group.*

(i) *If $w(G) = Z_2(G)$, then $w(G/Z(G)) = Z(G/Z(G))$.*
(ii) *If $Exp(G) = p$, then $N(G) = w(G) = Z(G)$.*

Conjecture: *Let G be a finite p-group with $w(G) = Z_2(G)$. Then $[Z_2(G) : Z(G)] = p$ for $p \geq 7$.*

By using the methods described in **"Algorithms"** and the available classification [24] of groups of order dividing p^6, we classify the groups G of order dividing p^4 (up to isomorphism) with $w(G) = Z_2(G)$ for all primes and those of order dividing p^6 for a variety of primes. For $n \in \mathbb{N}$ let $f(n)$ denote the number of non-abelian groups G (up to isomorphism) of order n with $w(G) = Z_2(G)$. We prove the following.

Theorem: *Let G be a non-abelian p-group. Then*

(i) $f(2^3) = 1$ *and* $f(p^3) = 0$ *for all primes* $p > 2$.
(ii) $f(2^4) = 2$ *and* $f(p^4) = 1$ *for all primes* $p > 2$.
(iii) $f(2^5) = f(3^5) = 5$.
(iv) $f(2^6) = 12$, $f(3^6) = 13$ *and* $f(5^6) = 25$.

Based on the experimental evidence we give a conjectural description of $f(p^5)$ for $p \geq 5$, and $f(p^6)$ for $p \geq 7$. It is interesting to note that these appear to be PORC polynomials, see [20]. The number of all p-groups of order p^5 and p^6 is known to be a PORC polynomial, but of significantly larger degree than $f(p^5)$ and $f(p^6)$.

Conjecture: *Let G be a non-abelian p-group. Then*

(i) $f(p^5) = 4 + \gcd(p-1, 4)$ *for all primes* $p \geq 5$.
(ii) $f(p^6) = 11 + 2(\gcd(p-1, 3) + \gcd(p-1, 4) + \gcd(p-1, 5))$ *for all primes* $p \geq 7$.

7.2 Zusammenfassung und Ausblick

Die Wielandt-Untergruppe $w(G)$ und ihre Verallgemeinerungen sind gruppentheoretische Konzepte, die zuerst als Verallgemeinerungen des Zentrums einer Gruppe G eingeführt wurden. Die erste Idee in dieser Richtung ist die von Baer 1935 eingeführte Norm $N(G)$. Der Vorteil der Wielandt-Untergruppe gegenüber der Norm ist, dass sie für eine deutlich größere Klassen von Gruppen nicht trivial ist.

In meiner MPhil-Arbeit [2], siehe auch [3], wird eine Verallgemeinerung der Wielandt-Untergruppe eingeführt. Sei G eine Gruppe und N eine normale Untergruppe von G. Die verallgemeinerte Wielandt-Untergruppe $w_N(G)$ von G bezüglich N ist der Schnitt der Normalisatoren der subnormalen, in N enthaltenen Untergruppen von G. Das erste Ziel dieser Arbeit ist es, die Struktur dieser verallgemeinerten Wielandt-Untergruppe zu untersuchen. Der Vollständigkeit halber wird eine Variation der Ergebnisse aus [3] wiederholt und ein Fehler in [3] korrigiert.

Wir führen eine neue Verallgemeinerung der Wielandt-Untergruppe ein, welche wir "relative Wielandt-Untergruppe" nennen. Die relative Wielandt-Untergruppe $w_c(G)$ von G ist der Schnitt der Normalisatoren der zyklischen subnormalen Untergruppen von G. Wir bemerken, dass $w_c(G)$ einige Eigenschaften erfüllt, die im Fall der gewöhnlichen Wielandt-Untergruppe nicht erfüllt sind. Die Gruppe $w_c(G)$ ist, verglichen mit $w(G)$, für eine größere Klassen von Gruppen nicht-trivial. Hierfür geben wir ein Beispiel.

Wir entwickeln Algorithmen um die Norm, die Wielandt-Untergruppe und ihre Verallgemeinerungen zu berechnen. Darauf basierend untersuchen wir die Gruppen, deren Ordnung p^6 für eine Primzahl p teilt und deren Wielandt-Untergruppe maximal ist.

Im Folgenden geben wir eine detailliertere Beschreibung der Ergebnisse dieser Arbeit.

Die verallgemeinerte Wielandt-Untergruppe und ihre Länge

Das folgende Theorem charakterisiert die Wielandt-Untergruppen höherer Ordnung und ihre Länge vollständig.

Theorem: *Sei G eine endliche Gruppe und $N \trianglelefteq G$.*

(i) *Sei $i \geq 0$, dann gilt $w_{N,i+1} = \cap\{N_G(K) \mid w_{N,i}(G) \leq K \leq Nw_{N,i}(G), K \lhd\lhd G\}$.*

(ii) *Seien $m, n \geq 0$ ganze Zahlen. Wenn die verallgemeinerte Wielandt-Länge von $G/w_{N,m}(G)$ bezüglich $Nw_{N,m}(G)/w_{N,m}(G)$ gleich n ist, dann gilt $wl_N(G) = m + n$.*

Das folgende Theorem gibt eine Strukturbeschreibung der auflösbaren Gruppen mit verallgemeinerter Wielandt-Länge eins.

Theorem: *Jede auflösbare Gruppe G mit $wl_{Fit(G)}(G) = 1$ ist metabelsch.*

Die relative Wielandt-Untergruppe

Wir definieren und untersuchen die relative Wielandt-Untergruppe $w_c(G)$. Für alle endlichen Gruppen G ist $w_c(G)$ nicht-trivial, da für eine beliebige Gruppe $N(G) \leq w(G) \leq w_c(G)$ gilt und $w(G)$ nicht-trivial ist. Im folgenden Theorem zeigen wir einige Eigenschaften der relativen Wielandt-Untergruppe für polyzyklische Gruppen.

Theorem: *Sei G eine polyzyklische Gruppe.*

(i) $w_c(G)Z(Fit(G))/Z(Fit(G)) \cong w_c(G)/w_c(G) \cap Z(Fit(G))$ *ist endlich und abelsch.*

(ii) $w_c(G)$ *ist überauflösbar und metabelsch.*

Die Gleichheit im nächsten Theorem gilt nicht für die gewöhnliche Wielandt-Untergruppe und ihre lokale Variante.

Theorem: *Seien G und H Gruppen. Dann gilt:*

(i) $w_c(G \times H) = w_c(G) \times w_c(H)$.

(ii) $w_c^p(G \times H) = w_c^p(G) \times w_c^p(H)$.

Beispiel: *Sei D_∞ die unendliche Diedergruppe. Dann ist $w(D_\infty) = \{1\}$ und $w_c(D_\infty) = D_\infty$. Das heißt, die gewöhnliche Wielandt-Untergruppe und die relative Wielandt-Untergruppe können sich unterscheiden.*

Algorithmen

Wir haben Algorithmen entwickelt, um die Norm einer Gruppe G für endliche Gruppen G und polyzyklische Gruppen G zu berechnen. Wir haben außerdem Methoden entwickelt, um die gewöhnliche, die verallgemeinerte und die relative Wielandt-Untergruppe für endliche Gruppen zu berechnen. Wir bemerken, dass es momentan keine Methode gibt, um die Wielandt-Untergruppe für eine beliebige, unendliche polyzyklische Gruppe zu berechnen. Weiterhin geben wir für endliche Gruppen eine Methode an, um zu entscheiden, ob die gegebene Gruppe eine T-Gruppe ist oder nicht.

p-Gruppen mit maximaler Wielandt-Untergruppe

Eine endliche p-Gruppe G ist nilpotent und daher gilt nach Schenkman [32] $Z(G) \leq w(G) = N(G) \leq Z_2(G)$. Deshalb scheint es interessant zu sein, den Extremfall der p-Gruppen G mit $w(G) = Z_2(G)$ zu verstehen. Wir geben im folgenden eine Teilantwort auf eine Frage von Guo [19] über die Struktur der Gruppen mit $w(G) = Z_2(G)$.

Theorem: *Sei G eine endliche p-Gruppe.*

(i) *Wenn* $w(G) = Z_2(G)$ *gilt, dann ist* $w(G/Z(G)) = Z(G/Z/G))$.

(ii) *Ist $Exp(G) = p$, dann gilt $N(G) = w(G) = Z(G)$.*

Vermutung: *Sei G eine endliche p-Gruppe mit $w(G) = Z_2(G)$. Dann gilt $[Z_2(G) : Z(G)] = p$ für $p \geq 7$.*

Wir klassifizieren die Gruppen mit maximaler Wielandt-Untergruppe, deren Ordnung p^4 teilt (bis auf Isomorphie) für alle Primzahlen und diejenigen, deren Ordnung p^6 teilt, für einige Primzahlen. Hierzu verwenden wir die Algorithmen und die vorhandenen Klassifikation der Gruppen, deren Ordnung p^6 teilt [24]. Für $n \in \mathbb{N}$ bezeichne $f(n)$ die Anzahl der nicht abelschen Gruppen G der Ordnung n (bis auf Isomorphie) mit $w(G) = Z_2(G)$. Wir beweisen das folgende Theorem.

Theorem: *Sei G eine nicht abelsche p-Gruppe. Dann gilt:*

(i) $f(2^3) = 1$ *und* $f(p^3) = 0$ *für alle Primzahlen* $p > 2$.
(ii) $f(2^4) = 2$ *und* $f(p^4) = 1$ *für alle Primzahlen* $p > 2$.
(iii) $f(2^5) = f(3^5) = 5$.
(iv) $f(2^6) = 12$, $f(3^6) = 13$ *und* $f(5^6) = 25$.

Basierend auf unseren experimentellen Daten geben wir eine Vermutung für $f(p^5)$ für $p \geq 5$ und $f(p^6)$ für $p \geq 7$ an. Interessanterweise scheinen dies PORC-Polynome zu sein, siehe [20]. Die Anzahl aller p-Gruppen der Ordnungen p^5 und p^6 ist bekanntlich ein PORC-Polynom, allerdings von deutlich größerem Grad als $f(p^5)$ und $f(p^6)$.

Vermutung: *Sei G eine nicht abelsche p-Gruppe. Dann gilt:*

(i) $f(p^5) = 4 + \gcd(p-1, 4)$ *für alle Primzahlen* $p \geq 5$.
(ii) $f(p^6) = 11 + 2(\gcd(p-1, 3) + \gcd(p-1, 4) + \gcd(p-1, 5))$ *für alle Primzahlen* $p \geq 7$.

Bibliography

[1] A. Ali. On the Wielandt length of a finite supersoluble group. *Proc. Roy. Soc. Edinburgh Sect. A*, 130(6):1217–1226, 2000.

[2] F. Amin. Generalised Wielandt subgroup and its series. *MPhil Thesis, Quaid-i-Azam University Islamabad, Pakistan*, 2005.

[3] F. Amin, A. Ali, and M. Arif. On Generalized Wielandt Subgroup. *World Applied Sciences Journal*, 30(12):1939–1946, 2014.

[4] R. Baer. Der Kern, eine charakteristische Untergruppe. *Compositio Math.*, 1:254–283, 1935.

[5] R. Baer. Zentrum und Kern von Gruppen mit Elementen unendlicher Ordnung. *Compositio Math.*, 2:247–249, 1935.

[6] R. Baer. Norm and hypernorm. *Publ. Math. Debrecen*, 4:347–350, 1956.

[7] H. U. Besche, B. Eick, and E. A. O'Brien. A millennium project: constructing small groups. *Internat. J. Algebra Comput.*, 12(5):623–644, 2002.

[8] R. A. Bryce and J. Cossey. The Wielandt subgroup of a finite soluble group. *J. London Math. Soc. (2)*, 40(2):244–256, 1989.

[9] W. Burnside. *Theory of groups of finite order*. Dover Publications, Inc., New York, 1955. 2d ed.

[10] A. R. Camina. The Wielandt length of finite groups. *J. Algebra*, 15:142–148, 1970.

[11] A. Y. M. Chin and A. P. L. Lee. A characterization of higher order Wielandt subgroups and some applications. *Missouri J. Math. Sci.*, 21(3):206–209, 2009.

[12] J. Cossey. The Wielandt subgroup of a polycyclic group. *Glasgow Math. J.*, 33(2):231–234, 1991.

[13] B. Eick. Algorithms for polycyclic groups. *Habilitationsschrift, Universität Kassel*, 2001.

[14] B. Eick. Computing with infinite polycyclic groups. In *Groups and computation, III (Columbus, OH, 1999)*, volume 8 of *Ohio State Univ. Math. Res. Inst. Publ.*, pages 139–154. de Gruyter, Berlin, 2001.

[15] B. Eick. On the Fitting subgroup of a polycyclic-by-finite group and its applications. *J. Algebra*, 242(1):176–187, 2001.

[16] B. Eick, M. Horn, and W. Nickel. Polycyclic - a GAP package. Version 2.11, 2013.

[17] The GAP Group. *GAP – Groups, Algorithms, and Programming, Version 4.7.9*, 2015.

[18] W. Gaschütz. Gruppen, in denen das Normalteilersein transitiv ist. *J. Reine Angew. Math.*, 198:87–92, 1957.

[19] X. Guo. Power automorphisms and induced automorphisms in finite groups. In *Proceedings of the International Conference on Algebra 2010*, pages 321–331. World Sci. Publ., Hackensack, NJ, 2012.

[20] G. Higman. Enumerating p-groups. II. Problems whose solution is PORC. *Proc. London Math. Soc. (3)*, 10:566–582, 1960.

[21] D. F. Holt, B. Eick, and E. A. O'Brien. *Handbook of computational group theory.* Discrete Mathematics and its Applications (Boca Raton). Chapman & Hall/CRC, Boca Raton, FL, 2005.

[22] B. Huppert. *Endliche Gruppen. I.* Die Grundlehren der Mathematischen Wissenschaften, Band 134. Springer-Verlag, Berlin-New York, 1967.

[23] J. C. Lennox and S. E. Stonehewer. *Subnormal subgroups of groups.* Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 1987. Oxford Science Publications.

[24] M. F. Newman, E. A. O'Brien, and M. R. Vaughan-Lee. Groups and nilpotent Lie rings whose order is the sixth power of a prime. *J. Algebra*, 278(1):383–401, 2004.

[25] E. A. O'Brien and M. R. Vaughan-Lee. The groups with order p^7 for odd prime p. *J. Algebra*, 292(1):243–258, 2005.

[26] Y. J. Oh. Groups with irreducible representations of prime degree. *PhD Thesis, Dongguk University, Seoul*, 1985.

[27] T. A. Peng. Finite groups with pro-normal subgroups. *Proc. Amer. Math. Soc.*, 20:232–234, 1969.

[28] D. J. S. Robinson. Groups in which normality is a transitive relation. *Proc. Cambridge Philos. Soc.*, 60:21–38, 1964.

[29] D. J. S. Robinson. *A course in the theory of groups*, volume 80 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1996.

[30] J. S. Rose. *A course on group theory.* 1978.

[31] T. Sakamoto. Groups whose cyclic subnormal subgroups are normal. *Bull. Kyushu Inst. Technol. Pure Appl. Math.*, (48):15–19, 2001.

[32] E. Schenkman. On the norm of a group. *Illinois J. Math.*, 4:150–152, 1960.

[33] M. Vaughan-Lee and B. Eick. LiePRing-A GAP package for computing with nilpotent Lie rings of prime-power order. Version 1.8, 2014.

[34] H. Wielandt. Über den Normalisator der subnormalen Untergruppen. *Math. Z.*, 69:463–465, 1958.

[35] G. Zacher. Caratterizzazione die t-gruppi finiti risolubili. *Ricerche Mat.*, 1:287–294, 1952.

[36] X. Zhang and X. Guo. On the Wielandt subgroup in a p-group of maximal class. *Chin. Ann. Math. Ser. B*, 33(1):83–90, 2012.

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